

SPECTRAL ANALYSIS OF EVEN ORDER DIFFERENTIAL OPERATOR

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ABSTRACT. Using the method of similar operators, we study even order differential operator with periodic, semiperiodic and Dirichlet boundary conditions. We obtain the asymptotic formulas for eigenvalues of this operator, estimates for spectral decompositions and spectral projections. We also establish the asymptotic behavior of analytic semigroup of operators, where generator is minus operator under study.

1. INTRODUCTION AND THE MAIN RESULTS

Let $L_2[0, \omega]$ be the Hilbert space of square summable complex functions on $[0, \omega]$, $\omega > 0$, with inner product $(x, y) = \frac{1}{\omega} \int_0^\omega x(\tau) \overline{y(\tau)} d\tau$, $x, y \in L_2[0, \omega]$. By $W_2^{2k}[0, \omega]$, $k \geq 1$, we denote the Sobolev space $\{y \in L_2[0, \omega] \rightarrow \mathbb{C} : y \text{ has } 2k - 1 \text{ continuous derivative, } y^{(2k-1)} \text{ is absolutely continuous, } y^{(2k)} \in L_2[0, \omega]\}$.

The aim of this paper is to give a detailed spectral analysis for the operators $L_{bc} : D(L_{bc}) \subset L_2[0, \omega] \rightarrow L_2[0, \omega]$ defined by the following differential expression

$$l(y) = (-1)^k y^{(2k)} - qy \quad \text{with} \quad k \geq 1 \quad \text{and} \quad q \in L_2[0, \omega]$$

and the boundary conditions bc of the following types:

- (a) periodic $bc = per$: $y^{(j)}(0) = y^{(j)}(\omega)$, $j = 0, 1, \dots, 2k - 1$;
- (b) semiperiodic $bc = ap$: $y^{(j)}(0) = -y^{(j)}(\omega)$, $j = 0, 1, \dots, 2k - 1$;
- (c) Dirichlet $bc = dir$: $y(0) = y''(0) = \dots = y^{(2k-2)}(0) = 0$,
 $y(\omega) = y''(\omega) = \dots = y^{(2k-2)}(\omega) = 0$.

Thus, $D(L_{bc}) = \{y \in W_2^{2k}[0, \omega] : y \text{ satisfies conditions } bc\}$. The operators L_{bc} with the boundary conditions (a), (b) and (c) will be denoted by L_{per} , L_{ap} and L_{dir} , respectively.

The operator L_{bc} may be applied to investigate vibrations of beams, plates and shells (see [1]). For example (see [2, Chapter I.1.14]), the

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Vlasov model of the bending cylindrical shells gives vibration equations of the form $y^{(8)} + by = \lambda y$, where b is a potential and λ is a spectral parameter.

Many authors investigated even order differential operator. The asymptotic behavior of eigenvalues for this operator with regular and strongly regular boundary conditions and summable potential was established in [3, Chapter II, Theorem 2]. In [4] A. Badanin and E. Korotyaev carried out spectral analysis of even order differential operator in $L_2(\mathbb{R})$ with periodic potential from the space $L_1(\mathbb{T})$, where $\mathbb{T} = \mathbb{R} \setminus \mathbb{Z}$. They obtained many interesting results for this operator concerning description of spectrum, asymptotic behavior of eigenvalues and spectral gaps. In [5] similar results were obtained for the operator L_{per} with $k = 2$ and the potential q from the real space $L_1(\mathbb{T})$. Fourth order differential operator of a general form was considered in papers [6] – [9].

Asymptotic behavior of eigenvalues for the operator L_{bc} in negative Sobolev space $H^{-m}[-1, 1]$ with periodic and semiperiodic boundary conditions and potential q from $H^{-m}[-1, 1]$ were given in [10] – [12].

In [13] O.A. Veliev considered the nonself-adjoint ordinary differential operator with periodic and semiperiodic boundary conditions and summable complex-valued potentials. He obtained asymptotic formulas of eigenvalues and eigenfunctions. He also established necessary and sufficient conditions on the coefficient under which the root functions of these operators form Riesz basis in $L_2[0, 1]$.

In [14, Theorem 1] and [15] it was described the asymptotic behavior of eigenvalues for even order differential operator with the Dirichlet boundary conditions and for the operator L_{per} in the case $k = 2$, respectively.

A great number of papers is devoted to spectral analysis for the Hill operator with the Dirichlet boundary conditions. V.M. Marchenko [16, Theorem 1.5.1, Theorem 1.5.2] obtained the asymptotic representation of eigenvalues with periodic, semiperiodic and Dirichlet boundary conditions and potential q from the Sobolev space $H^n[0, 1]$, $n \geq 1$. Some important results for this operator were established by A.M Savchuk and A.A. Shkalikov in [17].

The asymptotic formula of eigenvalues for the Hill operator with singular potential and the Dirichlet boundary conditions was given in [18], [19, Theorem 0.3] and [20, Theorem 1.1].

In [21] F. Gestezy and V. Tkachenko established necessary and sufficient conditions on the coefficient under which the root functions of the Hill operator L_{bc} , $bc \in \{per, ap\}$, form Riesz and Schauder basis in $L_p[0, \pi]$, $p \in (1, \infty)$.

E.F. Akhmerova and Kh.Kh. Murtazin [22] considered the Hill operator with real potential q from $L_2[0, \pi]$. They established an asymptotic formula for eigenvalues and trace formulas.

In this paper we use a new approach to study the operators L_{bc} . It based on the method of similar operators developed in the general setting in [23] – [26]. This method allows to reduce the study of the general operator L_{bc} to an associated operator with a "simple" structure.

To state our main results, we need some notation.

The operator L_{bc} can be represented in the form $L_{bc} = L_{bc}^0 - Q$, where $L_{bc}^0 : D(L_{bc}^0) = D(L_{bc}) \subset L_2[0, \omega] \rightarrow L_2[0, \omega]$, $L_{bc}^0 y = (-1)^k y^{(2k)}$, $k \geq 1$, is an unperturbed operator and Q is the operator of multiplication by the potential q . As is well known, L_{bc}^0 is a self-adjoint operator with compact resolvent.

For $bc \in \{per, ap\}$ the operators L_{bc} and L_{bc}^0 will be denoted by L_θ and L_θ^0 , where $\theta = 0$ and $\theta = 1$ stands for $bc = per$ and $bc = ap$, respectively.

We study the operator L_{dir} for $k \geq 1$ and the operators L_{per} , L_{ap} for $k > 1$. For $k = 1$ the operators L_{per} and L_{ap} are the Hill operators with periodic and semiperiodic boundary conditions and have been considered in [27].

Now we describe the spectrum $\sigma(L_{bc}^0)$ and the eigenfunctions of the operators L_{per}^0 , L_{ap}^0 , L_{dir}^0 :

(a), (b): $\sigma(L_\theta^0) = \{\lambda_n, n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}\}$, where $\lambda_n = (\frac{\pi(2n+\theta)}{\omega})^{2k}$, $k > 1$; the corresponding eigenfunctions are $e_n(t) = e^{-i\frac{\pi(2n+\theta)}{\omega}t}$, $t \in [0, \omega]$. They form an orthonormal basis in $L_2[0, \omega]$;

(c): $\sigma(L_{dir}^0) = \{\lambda_{n,dir}, n \in \mathbb{N}\}$, where $\lambda_{n,dir} = (\frac{\pi n}{\omega})^{2k}$, $k \geq 1$; the corresponding eigenfunctions have the form $e_{n,dir}(t) = \sqrt{2} \sin \frac{\pi n}{\omega} t$, $t \in [0, \omega]$.

Since the potential q belongs to $L_2[0, \omega]$, then $q(t) = \sum_{l \in \mathbb{Z}} q_l e^{i\frac{2\pi l}{\omega}t}$. For the operator L_{dir}^0 we will use the following representation for the potential: $q(t) = \sqrt{2} \sum_{l=1}^{\infty} \tilde{q}_l \cos \frac{\pi l}{\omega} t$.

Note that the operator L_{per}^0 has double eigenvalues (except the eigenvalue $\lambda_0 = 0$). The operator L_{ap}^0 has double eigenvalues and the operator L_{dir}^0 has simple eigenvalues.

We will also use the following sequences:

$$(1.1) \quad \alpha(n) = \left(\frac{\|q\|_2^2}{n^2} + \sum_{\substack{|p| \leq n \\ p \neq 0}} \frac{|q_{p-n}|^2}{p^2} \right)^{\frac{1}{2}}, \quad n \in \mathbb{N},$$

for $bc \in \{per, ap\}$ and

$$(1.2) \quad \beta(n) = \left(\frac{\|q\|_2^2}{n^2} + \sum_{\substack{|p| \leq n \\ p \neq 0}} \frac{\tilde{q}(p, n)}{p^2} \right)^{\frac{1}{2}}, \quad n \in \mathbb{N},$$

for $bc = dir$. Here $\tilde{q}(p, n) = \max\{|\tilde{q}_{|p+n|}|^2, |\tilde{q}_{|p-n|}|^2\}$, $p \in \mathbb{Z}$, $n \in \mathbb{N}$. Note that these sequences are square summable.

Another sequence we need in what follows for $bc \in \{per, ap\}$ is

$$w_\theta(n) = \left(2 + \left| \frac{q_{-2n-\theta} + c_{12}(n)}{q_{2n+\theta} + c_{21}(n)} \right| + \left| \frac{q_{2n+\theta} + c_{21}(n)}{q_{-2n-\theta} + c_{12}(n)} \right| \right)^{\frac{1}{2}}, \quad n \in \mathbb{N},$$

where

$$(1.3) \quad c_{12}(n) = \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \in \mathbb{Z} \\ j \neq n, j \neq -n-\theta}} \frac{q_{-n-j-\theta} q_{j-n}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}},$$

$$(1.4) \quad c_{21}(n) = \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \in \mathbb{Z} \\ j \neq n, j \neq -n-\theta}} \frac{q_{n+j+\theta} q_{n-j}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}}.$$

Theorem 1.1. *The operators L_{bc} , $bc \in \{per, ap\}$, have compact resolvent and there exists $m \in \mathbb{Z}_+$ such that the spectrum $\sigma(L_{bc})$ has the form*

$$(1.5) \quad \sigma(L_{bc}) = \sigma_{(m)} \bigcup \left(\bigcup_{n \geq m+1} \sigma_n \right),$$

where $\sigma_{(m)}$ is a finite set with number of points not exceeding $2m+1$ and $\sigma_n = \{\tilde{\lambda}_n^-\} \cup \{\tilde{\lambda}_n^+\}$, $n \geq m+1$. The eigenvalues $\tilde{\lambda}_n^\mp$, $n \geq m+1$, have the following asymptotic representation

$$(1.6) \quad \begin{aligned} \tilde{\lambda}_n^\mp &= \left(\frac{\pi(2n+\theta)}{\omega} \right)^{2k} - q_0 - \frac{2\omega^{2k}}{\pi^{2k}} \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{q_{n-j} q_{j-n}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}} \mp \\ &\mp \left(q_{-2n-\theta} + \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \in \mathbb{Z} \\ j \neq n, j \neq -n-\theta}} \frac{q_{-n-j-\theta} q_{j-n}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}} \right)^{\frac{1}{2}} \left(q_{2n+\theta} + \right. \\ &\left. + \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \in \mathbb{Z} \\ j \neq n, j \neq -n-\theta}} \frac{q_{n+j+\theta} q_{n-j}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}} \right)^{\frac{1}{2}} + \xi_{bc}(n). \end{aligned}$$

Here the sequence $\xi_{bc} : m + \mathbb{N} \rightarrow (0, \infty)$ satisfies the estimate:

$$|\xi_{bc}(n)| \leq \frac{C_\theta}{n^{4k-3}} w_\theta(n) \alpha(2n + \theta)$$

for some constant $C_\theta > 0$.

Theorem 1.2. *Let q be a function of bounded variation; then the asymptotic representation (1.6) holds and, for some constant $\tilde{C}_\theta > 0$, the sequence ξ_{bc} satisfies the estimate:*

$$|\xi_{bc}(n)| \leq \frac{\tilde{C}_\theta}{n^{4k-2}}, \quad n \geq m + 1,$$

where m is defined in Theorem 1.1.

Corollary 1.3. *If the potential q belongs to $C^4[0, \omega]$, then the eigenvalues $\tilde{\lambda}_n^\mp$, $n \geq m + 1$, of the operator L_{per} have the following asymptotic formulas:*

$$\begin{aligned} \tilde{\lambda}_n^\mp &= \left(\frac{2\pi n}{\omega} \right)^{2k} - q_0 - \frac{2\omega^{2k}}{\pi^{2k}} \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{q_{n-j} q_{j-n}}{(2j)^{2k} - (2n)^{2k}} \mp \\ &\mp \left(q_{-2n} + \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \in \mathbb{Z} \\ j \neq \pm n}} \frac{q_{-n-j} q_{j-n}}{(2j)^{2k} - (2n)^{2k}} \right)^{\frac{1}{2}} \left(q_{2n} + \right. \\ &\left. + \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \in \mathbb{Z} \\ j \neq \pm n}} \frac{q_{n+j} q_{n-j}}{(2j)^{2k} - (2n)^{2k}} \right)^{\frac{1}{2}} + \mathcal{O}\left(\frac{1}{n^{4k-2}} \right). \end{aligned}$$

This corollary improves the corresponding result of H. Menken [15, Theorem 3.1].

Theorem 1.4. *Suppose $k > 1$. The operator L_{dir} has compact resolvent. There exists $m \in \mathbb{N}$ such that its spectrum is represented in the form (1.5), where $\sigma_{(m)}$ is a finite set with number of points not exceeding m and $\sigma_n = \{\tilde{\lambda}_{n,dir}\}$, $n \geq m + 1$. For the eigenvalues $\tilde{\lambda}_{n,dir}$, $n \geq m + 1$, we have the following representation*

$$\begin{aligned} (1.7) \quad \tilde{\lambda}_{n,dir} &= \left(\frac{\pi n}{\omega} \right)^{2k} - \frac{1}{\omega} \int_0^\omega q(t) dt + \frac{1}{\omega} \int_0^\omega q(t) \cos \frac{2\pi n}{\omega} t dt - \\ &- \frac{\omega^{2k}}{2\pi^{2k}} \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{(\tilde{q}_{|n-j|} - \tilde{q}_{n+j})^2}{j^{2k} - n^{2k}} + \eta_{dir}(n), \quad n \geq m + 1, \end{aligned}$$

where the sequence $\eta_{dir} : \{n \in \mathbb{N} \mid n \geq m + 1\} \rightarrow (0, \infty)$ satisfies the estimate:

$$|\eta_{dir}(n)| \leq \frac{M}{n^{4k-3}} \beta(2n)$$

for some constant $M > 0$.

If q is a function of bounded variation, then we have

$$\begin{aligned} \tilde{\lambda}_{n,dir} = & \left(\frac{\pi n}{\omega} \right)^{2k} - \frac{1}{\omega} \int_0^\omega q(t) dt + \frac{1}{\omega} \int_0^\omega q(t) \cos \frac{2\pi n}{\omega} t dt - \\ & - \frac{\omega^{2k}}{2\pi^{2k}} \sum_{\substack{j=1 \\ j \neq n}}^\infty \frac{(\tilde{q}_{|n-j|} - \tilde{q}_{n+j})^2}{j^{2k} - n^{2k}} + \mathcal{O}\left(\frac{1}{n^{4k-2}}\right), \quad n \geq m + 1. \end{aligned}$$

For $k > 1$ the estimates given in this theorem are stronger than the corresponding ones in [14, Theorem 1].

Theorem 1.5. For $k = 1$ the operator L_{dir} has compact resolvent and there exists $m \in \mathbb{N}$ such that its spectrum has the form (1.5), where $\sigma_{(m)}$ is a finite set with number of points not exceeding m and $\sigma_n = \{\tilde{\lambda}_{n,dir}\}$, $n \geq m + 1$. The eigenvalues $\tilde{\lambda}_{n,dir}$, $n \geq m + 1$, have the following form (1.8)

$$\tilde{\lambda}_{n,dir} = \left(\frac{\pi n}{\omega} \right)^2 - \frac{1}{\omega} \int_0^\omega q(t) dt + \frac{1}{\omega} \int_0^\omega q(t) \cos \frac{2\pi n}{\omega} t dt + \zeta(n), \quad n \geq m + 1,$$

where sequence $\zeta : \{n \in \mathbb{N} \mid n \geq m + 1\} \rightarrow (0, \infty)$ satisfies the estimates:

$$|\zeta(n)| \leq \frac{\widetilde{M}}{n} \beta(2n),$$

for some constant $\widetilde{M} > 0$.

If q is a function of bounded variation, then we have

$$\tilde{\lambda}_{n,dir} = \left(\frac{\pi n}{\omega} \right)^2 - \frac{1}{\omega} \int_0^\omega q(t) dt + \frac{1}{\omega} \int_0^\omega q(t) \cos \frac{2\pi n}{\omega} t dt + \mathcal{O}\left(\frac{1}{n^2}\right).$$

The results of this theorem are better than earlier ones for the Hill operator with the Dirichlet boundary conditions (cf. [14, Theorem 1], [19, Theorem 0.3], [20, Theorem 1.2], [22, Theorem 1]).

Corollary 1.6. The operator L_{dir} is a spectral operator (in the Dunford sense) (see [28]).

Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator with compact resolvent acting in complex Hilbert space \mathcal{H} . Its spectrum can be represented in

the form

$$(1.9) \quad \sigma(\mathcal{A}) = \bigcup_{n \geq 1} \Delta_n,$$

where Δ_n , $n \geq 1$, are mutually disjoint compact sets. Denote by $P(\Delta_n, \mathcal{A})$ the Riesz projection constructed for the spectral set Δ_n . Let Ω be an arbitrary subset of \mathbb{N} and $\Delta = \Delta(\Omega) = \bigcup_{s \in \Omega} \Delta_s$. By $P(\Delta, \mathcal{A})$ we denote the Riesz projection constructed for the spectral set Δ . Thus, $P(\Delta, \mathcal{A}) = \sum_{s \in \Omega} P(\Delta_s, \mathcal{A})$.

Definition 1.7 ([26]). The operator \mathcal{A} is called *spectral with respect to decomposition* (1.9) if the series $\sum_{n=1}^{\infty} P(\Delta_n, \mathcal{A})x$ converges unconditionally to x for every $x \in \mathcal{H}$.

Now, we return to the operators L_{bc} , $bc \in \{per, ap, dir\}$. By \mathbb{P}_n , $n \in \mathbb{Z}_+$, denote the Riesz projections constructed for sets $\{(\frac{\pi(2n+\theta)}{\omega})^{2k}\}$, $k > 1$. Next, by $P_{n,dir}$, $n \in \mathbb{N}$, denote the Riesz projections constructed for $\{(\frac{\pi n}{\omega})^{2k}\}$, $k \geq 1$. Hence, $L_{bc}^0 \mathbb{P}_n = \lambda_n \mathbb{P}_n$, $n \in \mathbb{Z}_+$, for $bc \in \{per, ap\}$, and $L_{dir}^0 P_{n,dir} = \lambda_{n,dir} P_{n,dir}$, $n \in \mathbb{N}$. For all $x \in L_2[0, \omega]$ these projections have the following form:

- (a) $\mathbb{P}_n x = P_{-n} x + P_n x = (x, e_{-n}) e_{-n} + (x, e_n) e_n$, $n \in \mathbb{N}$, $\mathbb{P}_0 x = P_0 x = (x, e_0) e_0$;
- (b) $\mathbb{P}_n x = P_{-n-1} x + P_n x = (x, e_{-n-1}) e_{-n-1} + (x, e_n) e_n$, $n \in \mathbb{Z}_+$;
- (c) $P_{n,dir} x = (x, e_{n,dir}) e_{n,dir}$, $n \in \mathbb{N}$.

Let m be as in Theorem 1.1 or Theorem 1.4 (for $bc \in \{per, ap\}$ or $bc = dir$, respectively). We denote by $\mathbb{P}_{(m)}$ the projection $\sum_{s=0}^m \mathbb{P}_s$ for

$bc \in \{per, ap\}$ and by $P_{(m)}$ the projection $\sum_{s=1}^m P_{s,dir}$ for $bc = dir$.

Let $bc \in \{per, ap\}$ and $\Omega \subset \mathbb{Z}_+ \setminus \{0, \dots, m\}$. For the set $\Delta = \Delta(\Omega) = \{\lambda_n, n \in \Omega\}$ the Riesz projection $P(\Delta, L_{bc}^0)$ is defined as

$$P(\Delta, L_{bc}^0)x = \sum_{n \in \Omega} \mathbb{P}_n x, \quad x \in L_2[0, \omega].$$

Further, we consider the set $\tilde{\Delta} = \tilde{\Delta}(\Omega) = \bigcup_{n \in \Omega} \sigma_n$, where σ_n is defined in Theorem 1.1. By $\tilde{\mathbb{P}}_n$ we denote the Riesz projection constructed for σ_n , $n \geq m+1$. Then the projection $P(\tilde{\Delta}, L_{bc})$ is defined as

$$P(\tilde{\Delta}, L_{bc})x = \sum_{n \in \Omega} \tilde{\mathbb{P}}_n x, \quad x \in L_2[0, \omega].$$

Similarly, we define the Riesz projections $P(\Delta, L_{dir}^0)$, $P(\tilde{\Delta}, L_{dir})$:
 $P(\Delta, L_{dir}^0)x = \sum_{n \in \Omega} P_{n,dir}x$, $P(\tilde{\Delta}, L_{dir})x = \sum_{n \in \Omega} \tilde{P}_{n,dir}x$, $x \in L_2[0, \omega]$.

Here $\tilde{P}_{n,dir}$ is the Riesz projection constructed for the singleton set $\sigma_n = \{\tilde{\lambda}_{n,dir}\}$, $n \geq m+1$. In this case $\Omega \subset \mathbb{N} \setminus \{1, \dots, m\}$.

By $\|\cdot\|_2$ we denote the norm in the ideal $\mathfrak{S}_2(L_2[0, \omega])$ of the Hilbert-Schmidt operators (see [29, Chapter 3, Section 9] and Section 2 of the present paper).

Theorem 1.8. *Let m be as in Theorem 1.1 or in Theorem 1.4. For every subset $\Omega \subset \{m+1, m+2, \dots\}$ the following estimates hold:*

$$\|P(\tilde{\Delta}, L_{bc}) - P(\Delta, L_{bc}^0)\|_2 \leq \frac{M_1}{d^{2k-\frac{3}{2}}(\Omega)}.$$

Moreover, there exists a Hilbert-Schmidt operator \mathcal{U} with the norm $\|\mathcal{U}\|_2 \leq \frac{1}{2}$ such that

$$(1.10) \quad \|P(\tilde{\Delta}, L_{bc}) - (I + \mathcal{U})^{-1}P(\Delta, L_{bc}^0)(I + \mathcal{U})\|_2 \leq \frac{M_2}{d^{2k-1}(\Omega)}.$$

Here $M_1, M_2 > 0$ are some constants and $d(\Omega) = \min_{n \in \Omega} n$.

Note that the concrete form of the operator \mathcal{U} will be described in the proof of this theorem.

Corollary 1.9. *The following estimates of the spectral decompositions for the operators L_{bc} and L_{bc}^0 hold:*

$$\left\| P(\sigma_{(m)}, L_{bc}) + \sum_{k=m+1}^n \tilde{\mathbb{P}}_k - \sum_{k=0}^n \mathbb{P}_k \right\|_2 \leq \frac{M_3}{n^{2k-\frac{3}{2}}}, \quad n \geq m+1, \quad k > 1,$$

if $bc \in \{per, ap\}$, and

$$\left\| P(\sigma_{(m)}, L_{dir}) + \sum_{k=m+1}^n \tilde{P}_{k,dir} - \sum_{k=1}^n P_{k,dir} \right\|_2 \leq \frac{M_4}{n^{2k-\frac{3}{2}}}, \quad n \geq m+1, \quad k \geq 1,$$

where the set $\sigma_{(m)}$ is defined in (1.5) and $M_3, M_4 > 0$ are some constants.

In the following theorem we will use a sequence of operators $B_s \in \text{End } L_2[0, \omega]$, $s \geq m+1$. These operators are represented in the form $B_s = B_s^0 + B'_s$, where $B_s x = 0$ for all $x \in \text{Ker } \mathbb{P}_s$. The operators B_s^0 are defined as $B_s^0 e_{-s-\theta} = (q_{2s+\theta} + c_{21}(n))e_s$, $B_s^0 e_s = (q_{-2s-\theta} + c_{12}(n))e_{-s-\theta}$ and $B_s^0 x = 0$ for all $x \in \text{Ker } \mathbb{P}_s$. We do not know the form of these operators B'_s , $s \geq m+1$, but we obtain some estimates for their norms.

Theorem 1.10. *Let m be as in Theorem 1.1 or in Theorem 1.4. The operator $-L_{bc}$, $bc \in \{per, ap, dir\}$, is sectorial and it generates an analytical semigroup of operators $T : \mathbb{R}_+ \rightarrow \text{End } L_2[0, \omega]$. This semigroup is similar to a semigroup $\tilde{T} : \mathbb{R}_+ \rightarrow \text{End } L_2[0, \omega]$ of the following form*

$$\tilde{T}(t) = T_{(m)}(t) \oplus T^{(m)}(t), \quad t \in \mathbb{R}_+,$$

acting in $L_2[0, \omega] = \mathcal{H}_{(m)} \oplus \mathcal{H}^{(m)}$. Here $\mathcal{H}_{(m)} = \text{Im } \mathbb{P}_{(m)}$, $\mathcal{H}^{(m)} = \text{Im } (I - \mathbb{P}_{(m)})$ for $bc \in \{per, ap\}$ and $\mathcal{H}_{(m)} = \text{Im } P_{(m)}$, $\mathcal{H}^{(m)} = \text{Im } (I - P_{(m)})$ for $bc = dir$.

For $bc \in \{per, ap\}$ the semigroup $\tilde{T}^{(m)} : \mathbb{R}_+ \rightarrow \text{End } \mathcal{H}^{(m)}$ has the representation

$$(1.11) \quad \begin{aligned} \tilde{T}^{(m)}(t)x &= \sum_{s \geq m+1} e^{-t\pi^{2k}(2s+\theta)^{2k}/\omega^{2k} + q_0 t + t\omega^{2k}/\pi^{2k} \sum_{j \in \mathbb{Z}} \frac{q_s - j q_j - s}{(2j+\theta)^{2k} - (2s+\theta)^{2k}}} \\ &\cdot e^{a_s t} \left(\text{ch} b_s t + \frac{\text{sh} b_s t}{b_s} B_s \right) \mathbb{P}_s x, \quad x \in L_2[0, \omega]. \end{aligned}$$

Here

$$b_s = \sqrt{(q_{-2s-\theta} + c_{12}(n))(q_{2s+\theta} + c_{21}(n)) + \varepsilon_s},$$

where $c_{12}(n)$, $c_{21}(n)$ is defined in (1.3) and (1.4), respectively. The sequences of complex numbers a_s , ε_s , $s \geq m+1$, and sequences of operators B'_s , $s \geq m+1$, satisfy the estimate:

$$\max\{|a_s|, |\varepsilon_s|, \|B'_s\|_2\} \leq \frac{C_2}{s^{4k-3}} w_\theta(n) \alpha(2s + \theta), \quad s \geq m+1,$$

for some constant $C_2 > 0$.

For $bc = dir$ the semigroup $\tilde{T}^{(m)} : \mathbb{R}_+ \rightarrow \text{End } \mathcal{H}^{(m)}$ is represented as

$$\tilde{T}^{(m)}(t)x = \sum_{s \geq m+1} e^{-\tilde{\lambda}_{s, dir} t} P_{s, dir} x, \quad x \in L_2[0, \omega],$$

where the eigenvalues $\tilde{\lambda}_{s, dir}$, $s \geq m+1$, are defined in (1.7) for $k > 1$ and in (1.8) for $k = 1$.

Note that some of our results we stated above have been partially announced in [30].

The paper is organized as follows. In Section 2 we recall some basic notation for the method of similar operators. We also investigate abstract operators having similar properties to those of the operators L_{per} , L_{ap} and L_{dir} . In Section 3 we apply the results of Section 2 to the operator L_{bc} . In the final Section 4 we prove the main results.

2. CONSTRUCTION OF AN ADMISSIBLE TRIPLE

In this Section we apply the method of similar operators. We construct an admissible triple for abstract operators such that the spectral properties coincide with the spectral properties of operators L_{bc} , $bc \in \{per, ap, dir\}$. The main conception of the method of similar operators in a suitable form was introduced in [26]. In the present paper we will develop a new approach to this method.

First of all, recall the basic notions of the method of similar operators.

Let \mathcal{X} be a complex Banach space and let $\text{End } \mathcal{X}$ be a Banach algebra of bounded linear operators acting in \mathcal{X} . Let $A : D(A) \subset \mathcal{X} \rightarrow \mathcal{X}$ be a closed linear operator. By $\mathfrak{L}_A(\mathcal{X})$ we denote a Banach space of operators acting in \mathcal{X} , which are subordinated to operator A . The linear operator $X : D(X) \subset \mathcal{X} \rightarrow \mathcal{X}$ belongs to $\mathfrak{L}_A(\mathcal{X})$, if $D(A) \subseteq D(X)$ and $\|X\|_A = \inf\{C > 0 : \|Xx\| \leq C(\|x\| + \|Ax\|), x \in D(A)\} < \infty$. This is a norm in the space $\mathfrak{L}_A(\mathcal{X})$.

Definition 2.1. Two linear operators $A_i : D(A_i) \subset \mathcal{X} \rightarrow \mathcal{X}$, $i = 1, 2$, are called *similar*, if there exists a continuous an invertible operator $U \in \text{End } \mathcal{X}$ such that $UD(A_2) = D(A_1)$ and $A_1Ux = UA_2x$, $x \in D(A_2)$. The operator U is called a *transformation operator* of A_1 into A_2 .

The similar operators have some identical properties (see [26, Lemma 1]).

Definition 2.2 ([26]). Let \mathfrak{U} be a linear subspace of $\mathfrak{L}_A(\mathcal{X})$ and

$$J : \mathfrak{U} \rightarrow \mathfrak{U}, \quad \Gamma : \mathfrak{U} \rightarrow \text{End } \mathcal{X}$$

are linear transformers (i. e. linear operators in the space of linear operators). The triple $(\mathfrak{U}, J, \Gamma)$ is called *an admissible triple* for operator A and \mathfrak{U} is called a *space of admissible perturbations*, if the following conditions hold:

- 1) The space \mathfrak{U} is a Banach space with the norm $\|\cdot\|_*$ and it is embedded continuously to $\mathfrak{L}_A(\mathcal{X})$ (i. e. there exists a constant $C > 0$ such that $\|X\|_A \leq C\|X\|_*$ for every $X \in \mathfrak{U}$);
- 2) Operators J and Γ are continuously transformers and transformer J is a projection;
- 3) $(\Gamma X)D(A) \subset D(A)$ and

$$A\Gamma Xx - (\Gamma X)Ax = (X - JX)x, \quad X \in \mathfrak{U}, \quad x \in D(A);$$

4) $X\Gamma Y, (\Gamma X)Y \in \mathfrak{U}$ for every $X, Y \in \mathfrak{U}$ and there exists a constant $\gamma > 0$ such that

$$\|\Gamma\| \leq \gamma, \quad \max\{\|X\Gamma Y\|_*, \|(\Gamma X)Y\|_*\} \leq \gamma\|X\|_*\|Y\|_*;$$

5) for all $X \in \mathfrak{U}$ and $\varepsilon > 0$ there exists a number $\lambda_\varepsilon \in \rho(A)$ such that $\|X(A - \lambda_\varepsilon I)^{-1}\| < \varepsilon$, where I is the identity operator.

Theorem 2.3 ([26]). *Let $(\mathfrak{U}, J, \Gamma)$ be an admissible triple for the unperturbed operator $A : D(A) \subset \mathcal{X} \rightarrow \mathcal{X}$ and the operator B belongs to \mathfrak{U} . If*

$$\|J\|\|B\|_*\|\Gamma\| < \frac{1}{4},$$

then the operator $A - B$ is similar to $A - JX_$, where $X_* \in \mathfrak{U}$ is a solution of nonlinear equation*

$$X = B\Gamma X - (\Gamma X)(JB) - (\Gamma X)J(B\Gamma X) + B.$$

This solution can be found by the method of simple iterations setting $X_0 = 0$, $X_1 = B$, etc. The transformation of similarity of operator $A - B$ to $A - JX_$ is performed by invertible operator $I + \Gamma X_* \in \text{End } \mathcal{X}$, i. e.*

$$(2.1) \quad A - B = (I + \Gamma X_*)(A - JX_*)(I + \Gamma X_*)^{-1}.$$

In the following the space \mathcal{X} will be a complex Hilbert space \mathcal{H} . Let $\mathfrak{S}_2(\mathcal{H})$ be the Hilbert-Schmidt ideal of operators from algebra $\text{End } \mathcal{H}$ (see [29]). Recall this notion. By \mathbb{J} we denote one of the sets \mathbb{Z} or \mathbb{N} .

Definition 2.4. The operator $X : D(X) \subset \mathcal{H} \rightarrow \mathcal{H}$ is the Hilbert-Schmidt operator (i. e. operator X belongs to $\mathfrak{S}_2(\mathcal{H})$), if $\sum_{n,j \in \mathbb{J}} |(Xg_n, g_j)|^2$ is finite for some orthonormal basis $\{g_j, j \in \mathbb{J}\}$.

By $\|\cdot\|_2$ we denote the norm in $\mathfrak{S}_2(\mathcal{H})$: $\|X\|_2 = \left(\sum_{n,j \in \mathbb{J}} |(Xg_n, g_j)|^2 \right)^{\frac{1}{2}}$,

$X \in \mathfrak{S}_2(\mathcal{H})$.

In this paper we will use the Hilbert-Schmidt matrix (a_{nj}) , $n, j \in \mathbb{J}$, with the norm $\|(a_{nj})\|_2 = \left(\sum_{n,j \in \mathbb{J}} |a_{ij}|^2 \right)^{\frac{1}{2}}$. Note that for each operator $X \in \mathfrak{S}_2(\mathcal{H})$ its norm coincides with the norm of the Hilbert-Schmidt matrix $x_{nj} = (Xg_j, g_n)$, $n, j \in \mathbb{J}$, for every orthonormal basis $\{g_j, j \in \mathbb{J}\}$.

Further, we will use the following remarks (see [29]).

Remark 2.5. The operator $X \in \text{End } \mathcal{H}$ is the Hilbert-Schmidt operator iff its matrix $x_{nj} = (Xg_j, g_n)$, $n, j \in \mathbb{J}$, is the Hilbert-Schmidt matrix for some orthonormal basis $\{g_j, j \in \mathbb{J}\}$, and $\|X\|_2 = \|(x_{nj})\|_2$.

Remark 2.6. Let $\{Q_n, n \in \mathbb{J}\}$ be a system of orthoprojections which belong to $\text{End } \mathcal{H}$. For this system the following properties hold: 1) $Q_n Q_j = Q_j Q_n = 0$ for $n \neq j$; 2) $\sum_{n \in \mathbb{J}} Q_n x = x$ for every $x \in \mathcal{H}$. Then

$$\|X\|_2^2 = \sum_{n, j \in \mathbb{J}} \|Q_n X Q_j\|_2^2.$$

Remark 2.7. Let operator $X : D(X) \subset \mathcal{H} \rightarrow \mathcal{H}$ belongs to $\mathfrak{L}_A(\mathcal{H})$. If $\sum_{n, j \in \mathbb{J}} |(Xg_j, g_n)|^2 < \infty$ for some orthonormal basis $\{g_j, j \in \mathbb{J}\}$ with property $g_j \in D(X)$, $j \in \mathbb{J}$, then the operator X admits the unique extension to \mathcal{H} . It is a Hilbert-Schmidt operator denoted also by X .

We adapt the method of similar operators to abstract linear operators acting in the Hilbert space \mathcal{H} . Let $A_{bc} : D(A_{bc}) \subset \mathcal{H} \rightarrow \mathcal{H}$, $bc \in \{per, ap, dir\}$, be an unperturbed operator. Also we assume that a perturbation B belongs to the space $\mathfrak{S}_2(\mathcal{H})$.

Remark 2.8. We assume that spectral properties of operators A_{bc} coincide with spectral properties of operators L_{bc}^0 . We will use the same notation for eigenvalues, eigenfunctions and projections.

As above, for $bc \in \{per, ap\}$ the operator A_{bc} we will be denoted by A_θ , where $\theta = 0$ and $\theta = 1$ stands for $bc = per$ and $bc = ap$, respectively.

We consider operator $A_{bc} - B$, where A_{bc} is self-adjoint operator with compact resolvent. The spectrum $\sigma(A_{bc})$ consists of the following eigenvalues:

$$\lambda_n = \left(\frac{\pi}{\omega} (2n + \theta) \right)^{2k}, \quad k > 1, \quad \omega > 0, \quad n \in \mathbb{Z}_+, \quad \text{for } \theta \in \{0, 1\},$$

$$\lambda_{n, dir} = \left(\frac{\pi n}{\omega} \right)^{2k}, \quad k \geq 1, \quad \omega > 0, \quad n \in \mathbb{N}, \quad \text{for } bc = dir.$$

We assume that for $\theta = 0$ all its eigenvalues are double (except $\lambda_0 = 0$). For $\theta = 1$ all eigenvalues are double and for $bc = dir$ they are simple.

Let \mathbb{P}_n , $n \in \mathbb{Z}_+$, $\theta \in \{0, 1\}$, be an orthogonal Riesz projection constructed for the one point set $\{\lambda_n\}$ and let $P_{n, dir}$, $n \in \mathbb{N}$, be an orthogonal Riesz projection constructed for the one point set $\{\lambda_{n, dir}\}$. Hence, $A_\theta \mathbb{P}_n = \lambda_n \mathbb{P}_n$ and $A_{dir} P_{n, dir} = \lambda_{n, dir} P_{n, dir}$. For every $x \in \mathcal{H}$ these projections are defined as:

$$\mathbb{P}_n x = P_{-n} x + P_n x = (x, e_{-n}) e_{-n} + (x, e_n) e_n, \quad n \in \mathbb{N},$$

$$\mathbb{P}_0 x = P_0 x = (x, e_0) e_0, \quad bc = per,$$

$$\mathbb{P}_n x = P_{-n-1} x + P_n x = (x, e_{-n-1}) e_{-n-1} + (x, e_n) e_n, \quad n \in \mathbb{Z}_+, \quad bc = ap,$$

$$P_{n,dir}x = (x, e_{n,dir})e_{n,dir}, \quad n \in \mathbb{N}, \quad bc = dir.$$

Here e_n , $n \in \mathbb{Z}$, are the eigenfunctions of the operator A_{bc} for $bc \in \{per, ap\}$ and $e_{n,dir}$, $n \in \mathbb{N}$, are the eigenfunctions of the operator A_{dir} .

Let us start to construct an admissible triple for the operator A_{bc} , $bc \in \{per, ap, dir\}$. As space \mathfrak{U} we will use the Hilbert-Schmidt ideal $\mathfrak{S}_2(\mathcal{H})$.

Further, we construct the transformers J_{bc} , $\Gamma_{bc} \in \text{End } \mathfrak{S}_2(\mathcal{H})$, $bc \in \{per, ap, dir\}$. If $bc = per$, then

$$(2.2) \quad J_{per}X = \sum_{n=0}^{\infty} \mathbb{P}_n X \mathbb{P}_n = \sum_{n \in \mathbb{Z}} P_n X P_n + \sum_{n \in \mathbb{Z}} P_n X P_{-n} - P_0 X P_0, \quad X \in \mathfrak{S}_2(\mathcal{H}).$$

If $bc = ap$, then

$$(2.3) \quad J_{ap}X = \sum_{n=0}^{\infty} \mathbb{P}_n X \mathbb{P}_n = \sum_{n \in \mathbb{Z}} P_n X P_n + \sum_{n \in \mathbb{Z}} P_n X P_{-n-1}, \quad X \in \mathfrak{S}_2(\mathcal{H}).$$

Finally, for $bc = dir$ the transformer J_{dir} we define as

$$J_{dir}X = \sum_{n=1}^{\infty} P_{n,dir} X P_{n,dir}, \quad X \in \mathfrak{S}_2(\mathcal{H}).$$

The transformers Γ_{bc} , $bc \in \{per, ap, dir\}$, are determined by

$$\Gamma_{bc}X = \sum_{\substack{i,j=1 \\ \lambda_i \neq \lambda_j}}^{\infty} \frac{\mathbb{P}_i X \mathbb{P}_j}{\lambda_i - \lambda_j}, \quad \Gamma_{dir}X = \sum_{\substack{i,j=1 \\ \lambda_{i,dir} \neq \lambda_{j,dir}}}^{\infty} \frac{P_{i,dir} X P_{j,dir}}{\lambda_{i,dir} - \lambda_{j,dir}}, \quad X \in \mathfrak{S}_2(\mathcal{H}).$$

Since operator X belongs to $\mathfrak{S}_2(\mathcal{H})$, the values $\min_{\lambda_i \neq \lambda_j} |\lambda_i - \lambda_j|$ and $\min_{\substack{\lambda_{i,dir} \neq \lambda_{j,dir}}} |\lambda_{i,dir} - \lambda_{j,dir}|$ are positive. Next by remark 2.6 the transformer Γ_{bc} is well defined and bounded.

Also we consider the sequences of transformers $J_{m,bc}$, $\Gamma_{m,bc}$, $m \in \mathbb{Z}_+$, $bc \in \{per, ap\}$, defined by

$$J_{m,bc}X = J_{bc}(X - \mathbb{P}_{(m)}X\mathbb{P}_{(m)}) + \mathbb{P}_{(m)}X\mathbb{P}_{(m)}, \quad X \in \mathfrak{S}_2(\mathcal{H}),$$

$$\Gamma_{m,bc}X = \Gamma_{bc}(X - \mathbb{P}_{(m)}X\mathbb{P}_{(m)}), \quad X \in \mathfrak{S}_2(\mathcal{H}),$$

where $\mathbb{P}_{(m)} = \sum_{j=0}^m \mathbb{P}_j$. Note that $J_{bc} = J_{m,bc}$, $\Gamma_{bc} = \Gamma_{m,bc}$, if $m = 0$.

Let $m \in \mathbb{N}$. For $bc = dir$ these transformers are defined as

$$J_{m,dir}X = J_{dir}(X - P_{(m)}XP_{(m)}) + P_{(m)}XP_{(m)}, \quad X \in \mathfrak{S}_2(\mathcal{H}),$$

$$(2.4) \quad \Gamma_{m,dir}X = \Gamma_{dir}(X - P_{(m)}XP_{(m)}), \quad X \in \mathfrak{S}_2(\mathcal{H}),$$

where $P_{(m)} = \sum_{j=1}^m P_{j,dir}$.

Remark 2.9. Sometimes we will denote the operators J_{bc} , $J_{m,bc}$, Γ_{bc} , $\Gamma_{m,bc}$, $bc \in \{per, ap\}$, by J_θ , $J_{m,\theta}$, Γ_θ , $\Gamma_{m,\theta}$, $\theta \in \{0, 1\}$, respectively.

Lemma 2.10. *Transformers $J_{m,bc}$, $\Gamma_{m,bc}$, $bc \in \{per, ap, dir\}$, $m \in \mathbb{Z}_+$ (or $m \in \mathbb{N}$ for $bc = dir$), are self-adjoint operators in $\mathfrak{S}_2(\mathcal{H})$. Each transformer $J_{m,bc}$ is an orthogonal projection and transformer $\Gamma_{m,bc}$ is a compact operator. Moreover, the following estimates hold:*

$$(2.5) \quad \|\Gamma_{m,bc}\|_2 \leq \frac{\omega^{2k}}{4\pi^{2k}(2m+1)(2m+\theta)^{2k-2}}, \quad m \geq 0, \quad k > 1, \quad \theta \in \{0, 1\},$$

$$(2.6) \quad \|\Gamma_{m,dir}\|_2 \leq \frac{\omega^{2k}}{\pi^{2k}(2m+1)m^{2k-2}}, \quad m \geq 1, \quad k \geq 1, \quad bc = dir.$$

Proof. Let $bc \in \{per, ap\}$. For $i, j \in \mathbb{Z}$ with $\lambda_i \neq \lambda_j$, $|i - j| \geq m + 1$, the operators $\mathbb{P}_i X \mathbb{P}_j$, $X \in \mathfrak{S}_2(\mathcal{H})$, form the eigenspace, which corresponds the eigenvalue $(\lambda_i - \lambda_j)^{-1}$ of operator $\Gamma_{m,bc}$. These spaces are mutually orthogonal in $\mathfrak{S}_2(\mathcal{H})$. Hence, transformers $\Gamma_{m,bc}$, $m \geq 0$, are self-adjoint operators. Then $\|\Gamma_{m,bc}\|_2$ coincides with spectral radius $\max_{|i-j| \geq m+1} |\lambda_i - \lambda_j|^{-1}$. Therefore, inequalities (2.5) hold. Transformers $J_{m,bc}$, $bc \in \{per, ap\}$, $m \in \mathbb{Z}_+$, are orthogonal projections. This fact follows from definition of these operators. The case $bc = dir$ is considered in the same way. \square

Remark 2.11. Since the operator A_{bc} is self-adjoint operator, then the operator iA_{bc} generates strongly continuous group of isometries $T_{bc} : \mathbb{R} \rightarrow \text{End } \mathcal{H}$ (see the Stone's theorem [28]).

Consider an operator

$$i \text{ad}_{A_{bc}} : D(\text{ad}_{A_{bc}}) \subset \text{End } \mathcal{H} \rightarrow \text{End } \mathcal{H}$$

with domain $D(\text{ad}_{A_{bc}}) = \{X \in \text{End } \mathcal{H} : XD(A_{bc}) \subset D(A_{bc}) \text{ and an operator } A_{bc}X - XA_{bc} : D(A_{bc}) \subset \mathcal{H} \rightarrow \mathcal{H} \text{ can be extended on } \mathcal{H} \text{ to operator } Y \in \text{End } \mathcal{H} \text{ and } \text{ad}_{A_\theta}X = Y\}$. From [24] follows that the operator $i \text{ad}_{A_{bc}}$ generates group of isometries

$$\tilde{T}_{bc}(t)X = T_{bc}(t)XT_{bc}(-t), \quad X \in \text{End } \mathcal{H}.$$

This group is continuous in the strong operator topology. A restriction of an operators of this group on $\mathfrak{S}_2(\mathcal{H})$ is strongly continuous group of

isometries with generator $i \operatorname{ad}_{A_{bc}} : D(\operatorname{ad}_{A_{bc}}) \subset \mathfrak{S}_2(\mathcal{H}) \rightarrow \mathfrak{S}_2(\mathcal{H})$. His spectrum coincides with

$$i\{\lambda_n - \lambda_j; n, j \in \mathbb{Z}_+\} \text{ for } bc \in \{per, ap\} \text{ and } i\{\lambda_{n,dir} - \lambda_{j,dir}; n, j \in \mathbb{N}\}.$$

The transformer $J_{bc} \in \operatorname{End} \mathfrak{S}_2(\mathcal{H})$ is projection on subspace $\operatorname{Ker} \operatorname{ad}_{A_{bc}}$. Since the zero is isolated point of spectrum of operator $\operatorname{ad}_{A_{bc}}$, then the equation

$$\operatorname{ad}_{A_{bc}} X = Y - J_{bc} Y$$

has a solution for each operator Y from $\mathfrak{S}_2(\mathcal{H})$. Therefore, the operator $\operatorname{ad}_{A_{bc}}$ is invertible on subspace $\operatorname{Ker} J_{bc} = \operatorname{Im}(I - J_{bc})$. The inverse operator coincides with transformer Γ_{bc} on operators $\mathbb{P}_n X \mathbb{P}_j$, $X \in \mathfrak{S}_2(\mathcal{H})$, $\lambda_n \neq \lambda_j$, for $bc \in \{per, ap\}$ and $P_{n,dir} X P_{j,dir}$, $X \in \mathfrak{S}_2(\mathcal{H})$, $\lambda_{n,dir} \neq \lambda_{j,dir}$, for $bc = dir$. The linear combinations of these operators is dense in $\mathfrak{S}_2(\mathcal{H})$. Hence, the transformer Γ_{bc} is zero operator on $\operatorname{Im} J_{bc}$ and coincides with inverse to restriction $\operatorname{ad}_{A_{bc}}$ on $\operatorname{Im}(I - J_{bc})$.

Now we obtain some properties of operators $J_{m,bc}$, $\Gamma_{m,bc}$, $bc \in \{per, ap, dir\}$.

Lemma 2.12. *Transformers $J_{m,bc}$, $\Gamma_{m,bc} \in \operatorname{End} \mathfrak{S}_2(\mathcal{H})$, $m \in \mathbb{Z}_+$ (or $m \in \mathbb{N}$ for $bc = dir$), have the following properties:*

- 1) $P_{(m)}(J_{m,dir} X) = (J_{m,dir} X)P_{(m)} = P_{(m)} X P_{(m)}$, $X \in \mathfrak{S}_2(\mathcal{H})$;
- 2) $\mathbb{P}_{(m)}(J_{m,bc} X) = (J_{m,bc} X)\mathbb{P}_{(m)} = \mathbb{P}_{(m)} X \mathbb{P}_{(m)}$, $X \in \mathfrak{S}_2(\mathcal{H})$, for $bc \in \{per, ap\}$;
- 3) $J_{m,bc}((\Gamma_{m,bc} X)(J_{m,bc} Y)) = J_{m,bc}((J_{m,bc} X)(\Gamma_{m,bc} Y)) = 0$, $X, Y \in \mathfrak{S}_2(\mathcal{H})$, for $bc \in \{per, ap, dir\}$;
- 4) $\Gamma_{m,bc} X \in D(A_{bc})$ and $A_{bc}(\Gamma_{m,bc} X) - (\Gamma_{m,bc} X)A_{bc} = X - J_{m,bc} X$, $X \in \mathfrak{S}_2(\mathcal{H})$, for $bc \in \{per, ap, dir\}$;
- 5) for every operator $X \in \mathfrak{S}_2(\mathcal{H})$ the following estimates hold:

$$(2.7) \quad \|(\Gamma_{m,bc} X)\mathbb{P}_n\|_2 \leq \frac{\omega^{2k} \|X\mathbb{P}_n\|_2}{4\pi^{2k}(2n-1)(2n+\theta)^{2k-2}}, \quad n \geq m+1, \quad \theta \in \{0, 1\},$$

$$(2.8) \quad \|(\Gamma_{m,dir} X)P_{n,dir}\|_2 \leq \frac{\omega^{2k} \|XP_{n,dir}\|_2}{\pi^{2k}(2n-1)n^{2k-2}}, \quad n \geq m+1.$$

Proof. The properties 1), 2), 3) follow from properties of transformers $J_{m,bc}$, $\Gamma_{m,bc}$. Since the subspace $\operatorname{Im}(I - J_{m,bc})$ is an invariant subspace for transformer $\operatorname{ad}_{A_{bc}}$, then the property 4) holds.

Let us prove the property 5) for $bc \in \{per, ap\}$. Let $X_{sl} = \mathbb{P}_s X \mathbb{P}_l$, $s, l \in \mathbb{Z}_+$, be a block matrix of operator X . Then the matrix (Y_{sl}) of operator $Y = \Gamma_{m,bc} X$ has the following form

$$Y_{sl} = \frac{X_{sl}}{\lambda_s - \lambda_l}, \quad \max\{s, l\} \geq m+1, \quad Y_{sl} = 0, \quad \max\{s, l\} \leq m.$$

Hence,

$$\|(\Gamma_{m,bc}X)\mathbb{P}_n\|_2^2 = \sum_{\substack{s,l=0 \\ s \neq l}}^{\infty} \frac{\|\mathbb{P}_s(X\mathbb{P}_n)\mathbb{P}_l\|_2^2}{|\lambda_s - \lambda_l|^2} \leq \frac{\omega^{4k}\|X\mathbb{P}_n\|_2^2}{16\pi^{4k}(2n-1)^2(2n+\theta)^{4k-4}}.$$

The proof of inequality (2.8) is similar. \square

Lemma 2.13. *For all $m \in \mathbb{Z}_+$ for $bc \in \{per, ap\}$ (or $m \in \mathbb{N}$ for $bc = dir$) the triple $(\mathfrak{S}_2(\mathcal{H}), J_{m,bc}, \Gamma_{m,bc})$ is the admissible triple for operator A_{bc} . Constant $\gamma = \gamma(m)$ from definition 2.2 has the following estimates:*

$$\gamma(m) \leq \|\Gamma_{m,bc}\|_2 \leq \frac{\omega^{2k}}{4\pi^{2k}(2m+1)(2m+\theta)^{2k-2}}, \quad k > 1, \quad \theta \in \{0, 1\},$$

and

$$\gamma(m) \leq \|\Gamma_{m,dir}\|_2 \leq \frac{\omega^{2k}}{\pi^{2k}(2m+1)m^{2k-2}}, \quad k \geq 1.$$

Proof follows from lemmas 2.10, 2.12 and the inequality $\gamma(m) \leq \|\Gamma_{m,bc}\|_2$, $bc \in \{per, ap, dir\}$.

Theorem 2.14. *Let a number $m \in \mathbb{Z}_+$ for $bc \in \{per, ap\}$ (or $m \in \mathbb{N}$ for $bc = dir$) be such that*

$$(2.9) \quad \frac{\omega^{2k}\|B\|_2}{\pi^{2k}(2m+1)(2m+\theta)^{2k-2}} < 1, \quad k > 1, \quad \text{for } \theta \in \{0, 1\},$$

$$(2.10) \quad \frac{\omega^{2k}\|B\|_2}{\pi^{2k}(2m+1)m^{2k-2}} < \frac{1}{4}, \quad k \geq 1, \quad \text{for } bc = dir,$$

where the operator B belongs to $\mathfrak{S}_2(\mathcal{H})$. Then $A_{bc} - B$ is similar to $A_{bc} - J_{m,bc}X_*$, where $X_* \in \mathfrak{S}_2(\mathcal{H})$ is a solution of nonlinear equation (2.11)

$$X = B\Gamma_{m,bc}X - (\Gamma_{m,bc}X)(J_{m,bc}B) - (\Gamma_{m,bc}X)J_{m,bc}(B\Gamma_{m,bc}X) + B = \Phi(X).$$

This solution can be found by the method of simple iterations setting $X_0 = 0$, $X_1 = B, \dots$. Here the operator $\Phi : \mathfrak{S}_2(\mathcal{H}) \rightarrow \mathfrak{S}_2(\mathcal{H})$ is contraction operator in the ball $\{X \in \mathfrak{S}_2(\mathcal{H}) : \|X - B\|_2 \leq 3\|B\|_2\}$. The operator $I + \Gamma_{m,bc}X_*$ is transformation operator $A_{bc} - B$ to $A_{bc} - J_{m,bc}X_*$ and

$$A_{bc} - B = (I + \Gamma_{m,bc}X_*)(A_{bc} - J_{m,bc}X_*)(I + \Gamma_{m,bc}X_*)^{-1}.$$

Proof follows from Lemma 2.13 and Theorem 2.3.

Theorem 2.15. *Under the condition of Theorem 2.14, the operator $A_{bc} - B$ has compact resolvent and its spectrum coincides with spectrum of operators*

$$A_{bc} - J_{m,bc}X_* = A_{bc} - \mathbb{P}_{(m)}X_*\mathbb{P}_{(m)} - \sum_{n \geq m+1} \mathbb{P}_nX_*\mathbb{P}_n$$

for $bc \in \{per, ap\}$, and

$$A_{dir} - J_{m,dir}X_* = A_{dir} - P_{(m)}XP_{(m)} - \sum_{n \geq m+1} P_{n,dir}X_*P_{n,dir}$$

for $bc = dir$. Also we have

(2.12)

$$\sigma(A_{bc} - B) = \sigma(A_{(m)}) \bigcup \left(\bigcup_{n \geq m+1} \sigma(A_n) \right) = \sigma_{(m)} \bigcup \left(\bigcup_{n \geq m+1} \sigma_n \right),$$

where $A_{(m)}$ is a restriction of operator $A_{bc} - \mathbb{P}_{(m)}X_*\mathbb{P}_{(m)}$ on invariant subspaces $\mathcal{H}_{(m)} = \text{Im } \mathbb{P}_{(m)}$ for $bc \in \{per, ap\}$ or $\mathcal{H}_{(m)} = \text{Im } P_{(m)}$ for $bc = dir$. Operator A_n is a restriction of operator $A_{bc} - \mathbb{P}_nX_*\mathbb{P}_n$ on subspace $\mathcal{H}_n = \text{Im } \mathbb{P}_n$ if $bc \in \{per, ap\}$, or $\mathcal{H}_n = \text{Im } P_{n,dir}$ if $bc = dir$. Note that the sets $\sigma_{(m)}$ and σ_n , $n \geq m+1$, are mutually disjoint.

Proof. Obviously, operator $A_{bc} - J_{m,bc}X_*$ has compact resolvent. Therefore, operator $A_{bc} - B$ has compact resolvent and spectrums of these operators are coincide. The subspaces $\mathcal{H}_{(m)}$, \mathcal{H}_n are invariant with respect to operator $A_{bc} - J_{m,bc}X_*$. Hence, the right hand part of (2.12) is subset of $\sigma(A_{bc} - J_{m,bc}X_*) = \sigma(A_{bc} - B)$. The proof of reverse embedding can be found in [26, Section 4]. \square

We will use the following fact.

Lemma 2.16. *Suppose sequence of matrices \mathcal{A}_n , $n \in \mathbb{Z}_+$, has the form $\mathcal{A}_n = \mathcal{B}_n + \alpha_n \mathcal{C}_n$, where*

$$\mathcal{B}_n = \begin{pmatrix} b_{11}(n) & b_{12}(n) \\ b_{21}(n) & b_{22}(n) \end{pmatrix}, \quad \mathcal{C}_n = \begin{pmatrix} c_{11}(n) & c_{12}(n) \\ c_{21}(n) & c_{22}(n) \end{pmatrix}, \quad n \in \mathbb{Z}_+,$$

$b_{ij} \in l^2$, $c_{ij} \in l^1$, $1 \leq i, j \leq 2$, and sequence (α_n) converges to zero. Then eigenvalues $\mu^\mp(\mathcal{A}_n)$, $n \in \mathbb{Z}_+$, have the following representation

$$\mu^\mp(\mathcal{A}_n) = \mu^\mp(\mathcal{B}_n) + \alpha_n^{\frac{1}{2}} \gamma_n^\mp, \quad n \in \mathbb{Z}_+,$$

where $\mu^\mp(\mathcal{B}_n)$ is eigenvalues of matrix \mathcal{B}_n and sequence (γ_n^\mp) is summable with power $\frac{4}{3}$.

The proof is trivial.

Theorem 2.17. *Suppose $bc \in \{per, ap\}$, i. e. $\theta \in \{0, 1\}$. Under the conditions of Theorem 2.15, the sets σ_n , $n \geq m+1$, from representation (2.12) has number of points not exceeding two. Every set σ_n , $n \geq m+1$, coincides with spectrum of matrix \mathcal{A}_n . This matrix has the form*

$$(2.13) \quad \mathcal{A}_n = \left(\frac{\pi(2n + \theta)}{\omega} \right)^{2k} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \mathcal{B}_n^\theta + \mathcal{C}_n, \quad k > 1,$$

where $\mathcal{B}_n^\theta = \begin{pmatrix} (Be_{-n-\theta}, e_{-n-\theta}) & (Be_n, e_{-n-\theta}) \\ (Be_{-n-\theta}, e_n) & (Be_n, e_n) \end{pmatrix}$. The sequences of the Hilbert-Schmidt norms of matrix \mathcal{C}_n , $n \geq n_0$, satisfy the following estimates

$$\|\mathcal{C}_n\|_2 \leq \frac{\omega^{2k}}{2\pi^{2k}(2n-1)(2n+\theta)^{2k-2}} \|\mathbb{P}_n B - \mathbb{P}_n B \mathbb{P}_n\|_2 \|B \mathbb{P}_n - \mathbb{P}_n B \mathbb{P}_n\|_2, \\ k > 1, \quad n \geq n_0,$$

where $n_0 = \max\{m+1, \frac{1}{2}(\frac{3\|B\|_2\omega^{2k}}{2\pi^{2k}})^{\frac{1}{2k-1}}\}$. Eigenvalues $\tilde{\lambda}_n^\mp$ have the following form

$$(2.14) \quad \tilde{\lambda}_n^\mp = \left(\frac{\pi(2n + \theta)}{\omega} \right)^{2k} - \mu_n^\mp + \eta_n^\mp, \quad n \geq m+1,$$

where μ_n^\pm are eigenvalues of matrix \mathcal{B}_n^θ and sequences η_n^\mp , $n \geq m+1$, satisfy estimates $|\eta_n^\mp| \leq |\xi_n|/n^{2k-1}$, $n \geq m+1$. Here $(\xi_n) \in l^{\frac{4}{3}}$.

Proof. Applying projection \mathbb{P}_n to the left-hand and right-hand parts of equation (2.11) with $X = X_*$, we have

$$(2.15) \quad \mathbb{P}_n X_* \mathbb{P}_n = \mathbb{P}_n B \mathbb{P}_n + \mathbb{P}_n (B \Gamma_{m,\theta} X_*) \mathbb{P}_n, \quad n \geq m+1.$$

From Lemma 2.12, we have the following equalities for all operators $X, Y \in \mathfrak{S}_2(\mathcal{H})$:

- 1) $(J_{m,\theta} X) \mathbb{P}_n = \mathbb{P}_n (J_\theta X) \mathbb{P}_n = \mathbb{P}_n X \mathbb{P}_n$, $n \geq m+1$;
- 2) $\mathbb{P}_n (J_{m,\theta} X) (\Gamma_{m,\theta} Y) \mathbb{P}_n = 0$, $n \geq m+1$;
- 3) $\Gamma_{m,\theta} (\mathbb{P}_n X \mathbb{P}_n) = 0$, $n \geq m+1$;
- 4) $\mathbb{P}_n ((J_{m,\theta} X) \Gamma_{m,\theta} Y) \mathbb{P}_n = 0$, $n \geq m+1$.

The operator $\mathbb{P}_n (B \Gamma_{m,\theta} X_*) \mathbb{P}_n$ has the form

$$\begin{aligned} \mathbb{P}_n (B \Gamma_{m,\theta} X_*) \mathbb{P}_n &= \mathbb{P}_n (B - J_{m,\theta} B) (\Gamma_{m,\theta} X_*) \mathbb{P}_n = \\ &= \mathbb{P}_n (B - \mathbb{P}_n B \mathbb{P}_n) (\Gamma_{m,\theta} X_*) \mathbb{P}_n. \end{aligned}$$

Therefore,

$$(2.16) \quad \|\mathbb{P}_n (B \Gamma_{m,\theta} X_*) \mathbb{P}_n\|_2 \leq \|\mathbb{P}_n B - \mathbb{P}_n B \mathbb{P}_n\|_2 \|(\Gamma_{m,\theta} X_*) \mathbb{P}_n\|_2, \quad n \geq m+1.$$

Further, we obtain the estimates of sequence $\|(\Gamma_{m,\theta}X_*)\mathbb{P}_n\|_2$, $n \geq m+1$. Using equalities $(\Gamma_{m,\theta}X_*)\mathbb{P}_n = \Gamma_{m,\theta}(X_* - J_{m,\theta}X_*)\mathbb{P}_n$, $n \geq m+1$, we get

$$\|(\Gamma_{m,\theta}X_*)\mathbb{P}_n\|_2 \leq \frac{\|(X_* - J_{m,\theta}X_*)\mathbb{P}_n\|_2}{\min_{j \neq n} |\lambda_j - \lambda_n|} = \frac{\|X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n\|_2}{d_n},$$

where $d_n = \min_{j \neq n} |\lambda_j - \lambda_n| = \frac{4\pi^{2k}}{\omega^{2k}}(2n-1)(2n+\theta)^{2k-2}$.

Since operator X_* satisfies (2.11) and $\mathbb{P}_nX_*\mathbb{P}_n = \mathbb{P}_nB\mathbb{P}_n + \mathbb{P}_nB\Gamma_{m,\theta}(X_* - J_{m,\theta}X_*)\mathbb{P}_n$, $n \geq m+1$, we have

$$\begin{aligned} (X_* - J_{m,\theta}X_*)\mathbb{P}_n &= X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n = (B - \mathbb{P}_nB\mathbb{P}_n)\mathbb{P}_n + \\ &+ B\Gamma_{m,\theta}(X_* - \mathbb{P}_nX_*\mathbb{P}_n)\mathbb{P}_n - \\ &- \Gamma_{m,\theta}(X_* - \mathbb{P}_nX_*\mathbb{P}_n)\mathbb{P}_nX_*\mathbb{P}_n - \mathbb{P}_nB\Gamma_{m,\theta}(X_* - \mathbb{P}_nX_*\mathbb{P}_n)\mathbb{P}_n. \end{aligned}$$

Using inequality $\|\tilde{X}\|_2 \leq 4\|B\|_2$ (see Theorem 2.14), we get

$$\begin{aligned} \|X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n\|_2 &\leq \|B\mathbb{P}_n - \mathbb{P}_nB\mathbb{P}_n\|_2 + \frac{\|B\|_2\|X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n\|_2}{d_n} + \\ &+ \frac{4\|B\|_2\|X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n\|_2}{d_n} + \frac{\|B\|_2\|X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n\|_2}{d_n} = \\ &= \|B\mathbb{P}_n - \mathbb{P}_nB\mathbb{P}_n\|_2 + \frac{6\|B\|_2\|X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n\|_2}{d_n}. \end{aligned}$$

Thus, for $n \in \mathbb{N}$ with $\frac{6\|B\|_2}{d_n} \leq \frac{1}{2}$ we have

$$\|X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n\|_2 \leq 2\|B\mathbb{P}_n - \mathbb{P}_nB\mathbb{P}_n\|_2.$$

Hence,
(2.17)

$$\|(\Gamma_{m,\theta}X_*)\mathbb{P}_n\|_2 = \|\Gamma_{m,\theta}(X_*\mathbb{P}_n - \mathbb{P}_nX_*\mathbb{P}_n)\|_2 \leq \frac{2\|B\mathbb{P}_n - \mathbb{P}_nB\mathbb{P}_n\|_2}{d_n}.$$

Finally, using (2.16) and (2.17), for $n \geq n_0$ we get

$$\begin{aligned} \|\mathbb{P}_n(X_* - B)\mathbb{P}_n\|_2 &\leq \\ &\leq \frac{\omega^{2k}}{2\pi^{2k}(2n-1)(2n+\theta)^{2k-2}} \|\mathbb{P}_nB - \mathbb{P}_nB\mathbb{P}_n\|_2 \|B\mathbb{P}_n - \mathbb{P}_nB\mathbb{P}_n\|_2. \end{aligned}$$

If we consider the matrix of restrictions of operators from (2.15) on \mathcal{H}_n , then we get (2.13). From (2.13) and Lemma 2.16 we obtain the representation (2.14). \square

Theorem 2.18. *Spectrum of operator $A_{dir} - B$ has the form*

$$\sigma(A_{dir} - B) = \sigma_{(m)} \bigcup \left(\bigcup_{n \geq m+1} \{\tilde{\lambda}_{n,dir}\} \right),$$

where $\sigma_{(m)}$ is a finite set with number of points not exceeding $2m + 1$. The eigenvalues $\tilde{\lambda}_{n,dir}$, $n \geq n_1$, have the following asymptotic representation

$$\tilde{\lambda}_{n,dir} = \left(\frac{\pi n}{\omega} \right)^{2k} - (Be_{n,dir}, e_{n,dir}) + \eta_n, \quad k \geq 1, \quad n \geq n_1,$$

where $n_1 = \max\{m + 1, (\frac{6\|B\|_2\omega^{2k}}{\pi^{2k}})^{\frac{1}{2k-1}}\}$ and the sequence (η_n) satisfies the estimates

$$(2.18) \quad \begin{aligned} |\eta_n| &\leq \frac{2\omega^{2k}}{\pi^{2k}(2n-1)n^{2k-2}} \|P_{n,dir}B - P_{n,dir}BP_{n,dir}\|_2 \cdot \\ &\quad \cdot \|BP_{n,dir} - P_{n,dir}BP_{n,dir}\|_2. \end{aligned}$$

Proof is similar to the proof of Theorem 2.17 with $d_n = \min_{j \neq n} |\lambda_j - \lambda_n| \geq \frac{\pi^{2k}}{\omega^{2k}}(2n-1)n^{2k-2}$, $k \geq 1$.

Corollary 2.19. *Sequence $n \mapsto \|P_{n,dir}B - P_{n,dir}BP_{n,dir}\|_2 \|BP_{n,dir} - P_{n,dir}BP_{n,dir}\|_2 : \mathbb{N} \rightarrow \mathbb{R}_+$ is summable.*

Let $\text{Matr}_2(\mathbb{C})$ be an algebra of complex matrices 2×2 . A sequence (\mathcal{B}_n) of matrices $\mathcal{B}_n \in \text{Matr}_2(\mathbb{C})$ is called *stable* on the set $\Omega \subset \mathbb{Z}_+$, if there exists a sequence of matrices U_n , $n \in \Omega$, with the following properties:

1) there exists a finite subset $\Omega_0 \subset \Omega$ such that U_n , $n \in \Omega \setminus \Omega_0$, are invertible matrices and $\sup_{n \in \Omega \setminus \Omega_0} \|U_n\|_2 \|U_n^{-1}\|_2 = \widehat{M} < \infty$;

2) there exist sequences (μ_n^-) and (μ_n^+) such that:

$$\begin{pmatrix} \mu_n^+ & 0 \\ 0 & \mu_n^- \end{pmatrix} = U_n \mathcal{B}_n U_n^{-1}, \quad n \in \Omega \setminus \Omega_0.$$

Theorem 2.20. *If sequence (\mathcal{B}_n^θ) from representation (2.13) is stable on the set \mathbb{Z}_+ , then eigenvalues $\tilde{\lambda}_n^\mp$, $n \in \Omega$, $n \geq n_0$, of operator $A_\theta - B$, $\theta \in \{0, 1\}$, have the estimates:*

$$(2.19) \quad \begin{aligned} \left| \tilde{\lambda}_n^\mp - \left(\frac{\pi(2n + \theta)}{\omega} \right)^{2k} - \mu_n^\mp \right| &\leq \\ &\leq \frac{\widehat{M}\omega^{2k}}{2\pi^{2k}(2n-1)(2n+\theta)^{2k-2}} \|\mathbb{P}_n B - \mathbb{P}_n B \mathbb{P}_n\|_2 \|B \mathbb{P}_n - \mathbb{P}_n B \mathbb{P}_n\|_2. \end{aligned}$$

Proof. Multiplying both sides of (2.13) by U_n from the left and by U_n^{-1} from the right, we get the matrix $\begin{pmatrix} \mu_n^+ & 0 \\ 0 & \mu_n^- \end{pmatrix}$. This representation yields the estimates (2.19). \square

Now we obtain the estimates of spectral projections for the operator $A_{bc} - B$. We will use projections constructed for the unperturbed operator A_{bc} . We assume that conditions (2.9) and (2.10) of Theorem 2.14 hold.

Let Ω be an arbitrary subset from $\mathbb{Z}_+ \setminus \{0, \dots, m\}$. If $bc \in \{per, ap\}$, then

$$\mathbb{P}(\Omega) = \sum_{j \in \Omega} \mathbb{P}_j, \quad \tilde{\mathbb{P}}(\Omega) = \sum_{j \in \Omega} \tilde{\mathbb{P}}_j,$$

where $\tilde{\mathbb{P}}_j = P(\sigma_j, A_{bc} - B)$ is the Riesz projection constructed for spectral set σ_n , $n \geq m+1$, from representation (2.12) of spectrum of operator $A_{bc} - B$.

If $bc = dir$, then Ω will be considered as subset $\{n \in \mathbb{N} : n \geq m+1\}$. We put $P(\Omega) = \sum_{j \in \Omega} P_{j,dir}$, $\tilde{P}(\Omega) = \sum_{j \in \Omega} \tilde{P}_{j,dir}$.

Using [26, Lemma 1] and Theorems 2.14, 2.15, we obtain

$$\tilde{\mathbb{P}}(\Omega) = (I + \Gamma_{m,bc} X_*) \mathbb{P}(\Omega) (I + \Gamma_{m,bc} X_*)^{-1}, \quad bc \in \{per, ap\}.$$

Hence, the operator $\tilde{\mathbb{P}}(\Omega) - \mathbb{P}(\Omega)$ has the form

$$(2.20) \quad \tilde{\mathbb{P}}(\Omega) - \mathbb{P}(\Omega) = (\Gamma_{m,bc} X_* \mathbb{P}(\Omega) - \mathbb{P}(\Omega) \Gamma_{m,bc} X_*) (I + \Gamma_{m,bc} X_*)^{-1}.$$

The same representation holds for projections $P(\Omega)$.

In the next theorem by symbol $d(\Omega)$ we denote $\min_{n \in \Omega} n$.

Theorem 2.21. *We have the following estimates:*

$$(2.21) \quad \|\tilde{\mathbb{P}}(\Omega) - \mathbb{P}(\Omega)\|_2 \leq \frac{\widetilde{M}_1 \omega^{2k}}{4\pi^{2k} d^{2k-1}(\Omega)}, \quad n \geq m+1, \quad k > 1,$$

for $bc \in \{per, ap\}$ and

$$(2.22) \quad \|\tilde{P}(\Omega) - P(\Omega)\|_2 \leq \frac{\widetilde{M}_2 \omega^{2k}}{\pi^{2k} d^{2k-1}(\Omega)}, \quad n \geq m+1, \quad k \geq 1.$$

Here $\widetilde{M}_1, \widetilde{M}_2 > 0$ are constants which do not depend on $d(\Omega)$.

Proof. Suppose $bc \in \{per, ap\}$. Consider an arbitrary operator $X \in \mathfrak{S}_2(\mathcal{H})$. We estimate $\|(\Gamma_{m,\theta} X) \mathbb{P}(\Omega)\|_2$. By (2.7) and representation

$\mathbb{P}(\Omega) = \sum_{j \in \Omega} \mathbb{P}_j$ we obtain

$$\begin{aligned} \|(\Gamma_{m,\theta}X)\mathbb{P}(\Omega)\|_2^2 &\leq \sum_{n \in \Omega} \|(\Gamma_{m,\theta}X)\mathbb{P}_n\|_2^2 \leq \\ &\leq \sum_{n \in \Omega} \frac{\omega^{4k} \|X\mathbb{P}_n\|_2^2}{16\pi^{4k}(2n-1)^2(2n+\theta)^{4k-4}} \leq \frac{\omega^{4k} \|X\|_2^2}{16\pi^{4k}d^{4k-2}(\Omega)}. \end{aligned}$$

The same inequalities hold for the Hilbert-Schmidt norm of operator $\mathbb{P}(\Omega)\Gamma_{m,\theta}X$. Thus,

(2.23)

$$\|(\Gamma_{m,\theta}X)\mathbb{P}(\Omega)\|_2 \leq \frac{\omega^{2k} \|X\|_2}{4\pi^{2k}d^{2k-1}(\Omega)}, \quad \|\mathbb{P}(\Omega)\Gamma_{m,\theta}X\|_2 \leq \frac{\omega^{2k} \|X\|_2}{4\pi^{2k}d^{2k-1}(\Omega)}.$$

The estimates (2.21) follow by (2.20), (2.23) and condition $\|\Gamma_{m,\theta}X_*\|_2 < 1$. The case $bc = \text{dir}$ is proved in a similar way. \square

Theorem 2.22. *Under the conditions of Theorem 2.14, we get the following estimates for spectral decompositions:*

$$\|\widetilde{\mathbb{P}}_{(m)} + \sum_{j=m+1}^n \widetilde{\mathbb{P}}_j - \mathbb{P}_{(m)} - \sum_{j=m+1}^n \mathbb{P}_j\|_2 \leq \frac{\widetilde{M}_1 \omega^{2k}}{4\pi^{2k}n^{2k-1}}, \quad bc \in \{\text{per}, \text{ap}\},$$

and

$$\|\widetilde{P}_{(m)} + \sum_{j=m+1}^n \widetilde{P}_{j,\text{dir}} - P_{(m)} - \sum_{j=m+1}^n P_{j,\text{dir}}\|_2 \leq \frac{\widetilde{M}_2 \omega^{2k}}{\pi^{2k}n^{2k-1}}, \quad bc = \text{dir}.$$

Here $\widetilde{M}_1, \widetilde{M}_2 > 0$ are constants defined in Theorem 2.21.

3. PRELIMINARY TRANSFORMATION OF SIMILARITY FOR EVEN ORDER DIFFERENTIAL OPERATOR

In this section we apply the method of similar operators to operator $L_{bc} = L_{bc}^0 - Q$, $bc \in \{\text{per}, \text{ap}, \text{dir}\}$. Since the perturbation Q doesn't belong to $\mathfrak{S}_2(\mathcal{H})$, we have to make the preliminary transformation of similarity of operator L_{bc} to operator $L_{bc}^0 - B$, where $B \in \mathfrak{S}_2(\mathcal{H})$. We also obtain some estimates which we will use in the proof of the main results. Below by \mathcal{H} we denote the Hilbert space $L_2[0, \omega]$.

Remark 3.1. Without loss of generality we may assume $q_0 = 0$. But in the main theorems the value q_0 is involved.

Now we define the operators $J_{bc}Q, \Gamma_{bc}Q \in \mathfrak{S}_2(\mathcal{H})$, $bc \in \{\text{per}, \text{ap}, \text{dir}\}$. Here we will use the transformers $J_{bc}, \Gamma_{bc} \in \text{End } \mathfrak{S}_2(\mathcal{H})$, defined in Section 2.

Let $bc \in \{per, ap\}$. The operator $J_{bc}Q$ has the following form:

$$(3.1) \quad J_{bc}Q = \sum_{n=0}^{\infty} J_{bc}(Q\mathbb{P}_n) = \sum_{n=0}^{\infty} \mathbb{P}_n Q \mathbb{P}_n.$$

Since operators $\mathbb{P}_n Q \mathbb{P}_n$, $n \geq 0$, are the Hilbert-Schmidt operators, then J_{bc} is well defined. The matrix of a restriction of operator $\mathbb{P}_n Q \mathbb{P}_n$ on subspace $\text{Im } \mathbb{P}_n$ in basis $e_{-n-\theta}$, e_n has the form

$$\begin{pmatrix} 0 & q_{-2n-\theta} \\ q_{2n+\theta} & 0 \end{pmatrix}.$$

If $\theta = 0$, then subspace $\text{Im } \mathbb{P}_0$ is one-dimensional and $\mathbb{P}_0 Q \mathbb{P}_0 = q_0 \mathbb{P}_0 = 0$. Hence the series in (3.1) converge and

$$(3.2) \quad \|J_{bc}Q\|_2 = \|q\|_2.$$

In case $bc = dir$ the operator $J_{dir}Q$ is defined as

$$J_{dir}Q = \sum_{n=1}^{\infty} P_{n,dir} Q P_{n,dir} = \tilde{q}_0 I = 0.$$

Operators $J_{l,bc}Q$, $l \in \mathbb{Z}_+$, are determined by the following form

$$J_{l,bc}Q = J_{bc}Q - J_{bc}(\mathbb{P}_{(l)} Q \mathbb{P}_{(l)}) + \mathbb{P}_{(l)} Q \mathbb{P}_{(l)}, \quad bc \in \{per, ap\},$$

and

$$J_{l,dir}Q = J_{dir}Q - J_{dir}(P_{(l)} Q P_{(l)}) + P_{(l)} Q P_{(l)}.$$

The operator $\Gamma_{bc}Q$, $bc \in \{per, ap, dir\}$, we define as

$$\Gamma_{bc}Q = \sum_{\substack{i,j \in \mathbb{Z} \\ \lambda_i \neq \lambda_j}} \frac{P_i Q P_j}{\lambda_i - \lambda_j}, \quad bc \in \{per, ap\}, \quad \Gamma_{dir}Q = \sum_{\substack{i,j=1 \\ \lambda_i \neq \lambda_j}}^{\infty} \frac{P_{i,dir} Q P_{j,dir}}{\lambda_{i,dir} - \lambda_{j,dir}}.$$

Here $P_i Q P_j \in \mathfrak{S}_2(\mathcal{H})$, $i, j \in \mathbb{Z}$, and $P_{i,dir} Q P_{j,dir} \in \mathfrak{S}_2(\mathcal{H})$, $i, j \in \mathbb{N}$.

Further for case $bc = dir$ we compute the matrix coefficients q_{ij} , $i, j \in \mathbb{N}$, of the operator Q . We get

$$\begin{aligned} q_{ij} &= \frac{1}{\omega} \int_0^\omega q(t) e_i(t) \overline{e_j(t)} dt = \frac{2\sqrt{2}}{\omega} \sum_{l=1}^{\infty} \tilde{q}_l \int_0^\omega \cos \frac{\pi l}{\omega} t \sin \frac{\pi i}{\omega} t \sin \frac{\pi j}{\omega} t dt = \\ &= \frac{\sqrt{2}}{\omega} \sum_{l=1}^{\infty} \tilde{q}_l \int_0^\omega \cos \frac{\pi l}{\omega} t \left(\cos \frac{\pi}{\omega} (i-j)t - \cos \frac{\pi}{\omega} (i+j)t \right) dt = \\ &= \frac{\sqrt{2}}{\omega} (\tilde{q}_{|i-j|} - \tilde{q}_{i+j}) \int_0^\omega \cos^2 \frac{\pi l}{\omega} t dt = \frac{1}{\sqrt{2}} (\tilde{q}_{|i-j|} - \tilde{q}_{i+j}). \end{aligned}$$

Therefore,

$$(3.3) \quad q_{ij} = \frac{1}{\sqrt{2}}(\tilde{q}_{|i-j|} - \tilde{q}_{i+j}), \quad i, j \in \mathbb{N}.$$

We use matrix $\mathcal{Q} = (h_{ij})$, $i, j \in \mathbb{Z}$, to estimate the Hilbert-Schmidt norm of operator $\Gamma_{bc}Q$. If $bc \in \{per, ap\}$, then the matrix $\mathcal{Q} = (h_{ij})$, $i, j \in \mathbb{Z}$, for $k > 1$ has the form:

$$h_{ij} = \left(\frac{\omega^{2k}}{4\pi^{2k}} \frac{q_{i-j}}{(i-j)(i+j+\theta)((2i+\theta)^{2k-2} + (2i+\theta)^{2k-4}(2j+\theta)^2 + \dots + (2j+\theta)^{2k-2})} \right),$$

for $i \neq j$, $i \neq -j - \theta$, and $h_{ij} = 0$ for $i = j$, $i = -j - \theta$. If $bc = dir$, then the matrix $\mathcal{Q} = (h_{ij})$, $i, j \in \mathbb{N}$, is defined as:

$$h_{ij} = \begin{cases} \left(\frac{\omega^{2k}}{\pi^2 \sqrt{2}} \frac{\tilde{q}_{|i-j|} - \tilde{q}_{i+j}}{(i-j)(i+j)(i^{2k-2} + i^{2k-4}j^2 + \dots + j^{2k-2})} \right), & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases} \quad k > 1,$$

and

$$h_{ij} = \begin{cases} \left(\frac{\omega^2}{\pi^2 \sqrt{2}} \frac{\tilde{q}_{|i-j|} - \tilde{q}_{i+j}}{(i-j)(i+j)} \right), & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases} \quad k = 1.$$

Now we estimate the value $\|\mathbb{P}_n \Gamma_{bc}Q\|_2$, $n \in \mathbb{Z}_+$, $bc \in \{per, ap\}$. We have the following inequalities:

$$\begin{aligned} \|P_n \Gamma_{bc}Q\|_2^2 &= \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \cdot \sum_{\substack{p \neq n \\ p \neq -n-\theta}} \frac{|q_{n-p}|^2}{|n-p|^2 |n+p+\theta|^2 ((2n+\theta)^{2k-2} + \dots + (2p+\theta)^{2k-2})^2} \leq \\ &\leq \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \frac{1}{(2n+\theta)^{4k-4}} \sum_{\substack{j \neq 0 \\ j \neq -2n-\theta}} \frac{|q_j|^2}{j^2 |2n-j+\theta|^2} \leq \\ &\leq \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \frac{\|q\|_2^2}{(2n+\theta)^{4k-4} (2|n|-1)^2}, \quad n \in \mathbb{Z}. \end{aligned}$$

Therefore, for $n \in \mathbb{Z}_+$ we get:

$$(3.4) \quad \begin{aligned} \|\mathbb{P}_n(\Gamma_{bc}Q)\|_2^2 &\leq \|P_n \Gamma_{bc}Q\|_2^2 + \|P_{-n-\theta} \Gamma_{bc}Q\|_2^2 \leq \\ &\leq 2 \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \frac{\|q\|_2^2}{(2n+\theta)^{4k-4} (2n-1)^2}. \end{aligned}$$

Let us consider the case $bc = \text{dir}$. We estimate the value $\|P_{n,\text{dir}}(\Gamma_{\text{dir}}Q)\|_2$. Using (3.3), for $k > 1$ we have:

$$\begin{aligned}
 \|P_{n,\text{dir}}(\Gamma_{\text{dir}}Q)\|_2^2 &= \sum_{\substack{p=1 \\ p \neq n}}^{\infty} \frac{|q_{np}|^2}{|\lambda_{n,\text{dir}} - \lambda_{p,\text{dir}}|^2} = \frac{1}{2} \sum_{\substack{p=1 \\ p \neq n}}^{\infty} \frac{|\tilde{q}_{|n-p|} - \tilde{q}_{n+p}|^2}{|\lambda_{n,\text{dir}} - \lambda_{p,\text{dir}}|^2} \leq \\
 &\leq \left(\frac{\omega^{2k}}{\pi^{2k}}\right)^2 \sum_{\substack{p=1 \\ p \neq n}}^{\infty} \frac{|\tilde{q}_{|n-p|}|^2 + |\tilde{q}_{n+p}|^2}{|n-p|^2|n+p|^2|n^{2k-2} + \dots + p^{2k-2}|^2} \leq \\
 (3.5) \quad &\leq \left(\frac{\omega^{2k}}{\pi^{2k}}\right)^2 \frac{1}{n^{4k-4}} \sum_{\substack{p=1 \\ p \neq n}}^{\infty} \frac{|\tilde{q}_{|n-p|}|^2 + |\tilde{q}_{n+p}|^2}{|n-p|^2|n+p|^2} \leq 2 \left(\frac{\omega^{2k}}{\pi^{2k}}\right)^2 \frac{\|q\|_2^2}{n^{4k-4}(2n-1)^2}.
 \end{aligned}$$

Therefore,

$$(3.6) \quad \|P_{n,\text{dir}}(\Gamma_{\text{dir}}Q)\|_2 \leq \frac{\omega^{2k}\|q\|_2\sqrt{2}}{\pi^{2k}n^{2k-2}(2n-1)}.$$

If $k = 1$, then

$$\begin{aligned}
 \|P_{n,\text{dir}}(\Gamma_{\text{dir}}Q)\|_2^2 &= \frac{1}{2} \sum_{\substack{p=1 \\ p \neq n}}^{\infty} \frac{|\tilde{q}_{|n-p|} - \tilde{q}_{n+p}|^2}{|\lambda_{n,\text{dir}} - \lambda_{p,\text{dir}}|^2} \leq \\
 &\leq \left(\frac{\omega^2}{\pi^2}\right)^2 \sum_{\substack{p=1 \\ p \neq n}}^{\infty} \frac{|\tilde{q}_{|n-p|}|^2 + |\tilde{q}_{n+p}|^2}{|n^2 - p^2|^2} \leq 2 \left(\frac{\omega^2}{\pi^2}\right)^2 \frac{\|q\|_2^2}{(2n-1)^2}.
 \end{aligned}$$

Hence, in case $k = 1$ the formula (3.6) holds.

Now we define the operators $\Gamma_{l,bc}$, $bc \in \{\text{per}, \text{ap}, \text{dir}\}$, as:

$$(3.7) \quad \Gamma_{l,bc}Q = \Gamma_{bc}(Q - \mathbb{P}_{(l)}Q\mathbb{P}_{(l)}) = \Gamma_{bc}Q - \mathbb{P}_{(l)}(\Gamma_{bc}Q)\mathbb{P}_{(l)}, \quad l \in \mathbb{Z}_+,$$

for $bc \in \{\text{per}, \text{ap}\}$ and

$$(3.8) \quad \Gamma_{l,\text{dir}}Q = \Gamma_{\text{dir}}(Q - P_{(l)}QP_{(l)}) = \Gamma_{\text{dir}}Q - P_{(l)}(\Gamma_{\text{dir}}Q)P_{(l)}, \quad l \in \mathbb{N}.$$

We recall that $\mathbb{P}_{(l)} = \sum_{j=0}^l \mathbb{P}_j$ for $bc \in \{\text{per}, \text{ap}\}$ and $P_{(l)} = \sum_{j=1}^l P_{j,\text{dir}}$ for $bc = \text{dir}$.

Lemma 3.2. *The operators $\Gamma_{l,bc}Q$, $l \in \mathbb{Z}_+$, $bc \in \{\text{per}, \text{ap}\}$ (or $l \in \mathbb{N}$ in case $bc = \text{dir}$), are the Hilbert-Schmidt operators and $\lim_{l \rightarrow \infty} \|\Gamma_{l,bc}Q\|_2 = 0$.*

Proof. Let $bc \in \{per, ap\}$. Using (3.4), we obtain

$$\begin{aligned} \|\Gamma_{bc}Q\|_2^2 &= \sum_{n=0}^{\infty} \|\mathbb{P}_n(\Gamma_{bc}Q)\|_2^2 \leq \\ &\leq 2 \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \|q\|_2^2 \sum_{n=0}^{\infty} \frac{1}{(2n+\theta)^{4k-4}(2n-1)^2} < \infty. \end{aligned}$$

Now we consider the case $bc = dir$. From inequality (3.6) we get

$$\begin{aligned} \|\Gamma_{dir}Q\|_2^2 &= \sum_{n=1}^{\infty} \|P_{n,dir}(\Gamma_{dir}Q)\|_2^2 \leq \\ &\leq 2 \left(\frac{\omega^{2k}}{\pi^{2k}} \right)^2 \|q\|_2^2 \sum_{n=1}^{\infty} \frac{1}{n^{4k-4}(2n-1)^2} < \infty. \end{aligned}$$

Therefore, operators $\Gamma_{bc}Q$, $bc \in \{per, ap, dir\}$, are the Hilbert-Schmidt operators.

From formulas (3.7) and (3.8), we have $\lim_{l \rightarrow \infty} \|\Gamma_{l,bc}Q\|_2 = 0$, $bc \in \{per, ap, dir\}$. \square

We need some technical lemmas.

Lemma 3.3. *Operators $\Gamma_{l,bc}Q$, $l \in \mathbb{Z}_+$, for $bc \in \{per, ap\}$ (or $l \in \mathbb{N}$ for $bc = dir$) have the following properties:*

- 1) $(\Gamma_{l,bc}Q)D(L_{bc}^0) \subset D(L_{bc}^0)$;
- 2) $L_{bc}^0(\Gamma_{l,bc}Q)x - (\Gamma_{l,bc}Q)L_{bc}^0x = (Q - J_{l,bc}Q)x$, $x \in D(L_{bc}^0)$;
- 3) for all $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \rho(L_{bc}^0)$ such that $\|Q(L_{bc}^0 - \lambda_\varepsilon I)^{-1}\|_2 < \varepsilon$.

Proof. Let $\lambda_0 \in \rho(L_{bc}^0)$. We consider the sequence of projections $\mathcal{P}_{(n)} = \sum_{j=0}^n \mathbb{P}_j$ for $bc \in \{per, ap\}$ and $\mathcal{P}_{(n)} = \sum_{j=1}^n P_{j,dir}$ for $bc = dir$. For each $y \in \mathcal{H}$ we get

$$\begin{aligned} (3.9) \quad \mathcal{P}_{(n)}L_{bc}^0(\Gamma_{l,bc}Q)(L_{bc}^0 - \lambda_0 I)^{-1}y &= \mathcal{P}_{(n)}(\Gamma_{l,bc}Q)L_{bc}^0(L_{bc}^0 - \lambda_0 I)^{-1}y + \\ &+ \mathcal{P}_{(n)}(Q - J_{l,bc}Q)(L_{bc}^0 - \lambda_0 I)^{-1}y = \mathcal{P}_{(n)}\mathcal{L}y, \end{aligned}$$

where $\mathcal{L}y = (\Gamma_{l,bc}Q)L_{bc}^0(L_{bc}^0 - \lambda_0 I)^{-1}y + (Q - J_{l,bc}Q)(L_{bc}^0 - \lambda_0 I)^{-1}y$. This equality is verified for basis vectors in \mathcal{H} . The operator \mathcal{L} is the Hilbert-Schmidt operator. Hence, the sequence of operators from the right-hand side of (3.9) converges to \mathcal{L} in $\mathfrak{S}_2(\mathcal{H})$. Since the operator L_{bc}^0 is closed, then $\Gamma_{l,bc}Qx \in D(L_{bc}^0)$ for every $x \in D(L_{bc}^0)$ and

$$\begin{aligned} L_{bc}^0(\Gamma_{l,bc}Q)(L_{bc}^0 - \lambda_0 I)^{-1} &= (\Gamma_{l,bc}Q)L_{bc}^0(L_{bc}^0 - \lambda_0 I)^{-1} + \\ &+ (Q - J_{l,bc}Q)(L_{bc}^0 - \lambda_0 I)^{-1}. \end{aligned}$$

Therefore, we have proved the properties 1) and 2).

Now we prove the property 3). Suppose $bc = per$. Other cases are considered in a similar way. Complex numbers in , $n \in \mathbb{N}$, belong to the resolvent set of operator L_{per}^0 . The matrix of operator $Q(L_{per}^0 - inI)^{-1}$ has the following form

$$\left(\frac{q_{p-j}}{\frac{2^{2k}\pi^{2k}j^{2k}}{\omega^{2k}} - in} \right), \quad p, j \in \mathbb{Z}.$$

This matrix is the Hilbert-Schmidt matrix and

$$\begin{aligned} \|Q(L_{per}^0 - inI)^{-1}\|_2^2 &= \omega^{4k} \sum_{p,j \in \mathbb{Z}} \frac{|q_{p-j}|^2}{|2^{2k}\pi^{2k}j^{2k} - in\omega^{2k}|^2} = \\ &= \omega^{4k} \|q\|_2^2 \sum_{j \in \mathbb{Z}} \frac{1}{2^{4k}\pi^{4k}j^{4k} + n^2\omega^{4k}} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Therefore, property 3) holds. \square

Lemma 3.4. *Operators $Q\Gamma_{l,bc}Q$, $l \in \mathbb{Z}_+$, $bc \in \{per, ap\}$ (or $l \in \mathbb{N}$ for $bc = dir$), are the Hilbert-Schmidt operators.*

Proof. It is sufficient to prove the statement for the matrix $\mathcal{V} = (\tilde{q}_{pn})$, $p, n \in \mathbb{Z}$, of operator $Q\Gamma_{bc}Q : D(L_{bc}^0) \subset \mathcal{H} \rightarrow \mathcal{H}$. This fact follows from the definition of operator $\Gamma_{l,bc}Q$, $l \in \mathbb{Z}_+$, for $bc \in \{per, ap\}$ (or $l \in \mathbb{N}$ for $bc = dir$), and remark 2.7.

Suppose $bc = per$ and $k > 1$. Other cases for $k > 1$ are considered in a similar way. The matrix $\mathcal{V} = (\tilde{q}_{pn})$, $p, n \in \mathbb{Z}$, of the operator $Q\Gamma_{per}Q$ has the following form

$$\tilde{q}_{pn} = \frac{\omega^{2k}}{4\pi^{2k}} \sum_{\substack{j \in \mathbb{Z} \\ |j| \neq |n|}} \frac{q_{p-j}q_{j-n}}{(j-n)(j+n)((2j)^{2k-2} + \dots + (2n)^{2k-2})}, \quad p, n \in \mathbb{Z}.$$

Now we estimate the Hilbert-Schmidt norm $\|\mathcal{V}\|_2$ of the matrix $\mathcal{V} = (\tilde{q}_{pn})$, $p, n \in \mathbb{Z}$. Using the Hölder inequality, we get:

$$\begin{aligned} \|\mathcal{V}\|_2^2 &= \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \cdot \sum_{p,n \in \mathbb{Z}} \left| \sum_{\substack{j \in \mathbb{Z} \\ |j| \neq |n|}} \frac{q_{p-j}q_{j-n}}{(j-n)(j+n)((2j)^{2k-2} + \dots + (2n)^{2k-2})} \right|^2 = \\ &= \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \sum_{p,n \in \mathbb{Z}} \left| \sum_{\substack{l \neq 0 \\ l \neq -2n}} \frac{q_{p-n-l}q_l}{l(l+2n)((2(n+l))^{2k-2} + \dots + (2n)^{2k-2})} \right|^2 = \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \sum_{p,n \in \mathbb{Z}} \frac{1}{(2n)^{2k-2}} \sum_{\substack{l \neq 0 \\ l \neq -2n}} \frac{|q_l|^2}{l^2} \frac{|q_{p-n-l}|^2}{|l+2n|^2} \leq \\
&\leq \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \|q\|_2^2 \sum_{n \in \mathbb{Z}} \frac{1}{(2n)^{2k}} \sum_{l \neq 0} \frac{|q_l|^2}{l^2} \leq \\
&\leq \left(\frac{\omega^{2k}}{4\pi^{2k}} \right)^2 \|q\|_2^4 \frac{\pi^2}{3} \sum_{n \in \mathbb{Z}} \frac{1}{(2n)^{2k}} < \infty.
\end{aligned}$$

By remark 2.7, the operator $Q\Gamma_{per}Q$ has bounded expansion on the space \mathcal{H} to the Hilbert-Schmidt operator.

Now we prove this theorem for $bc = dir$ in case $k = 1$. The matrix $\mathcal{V} = (\Pi_{pn})$, $p, n \in \mathbb{N}$, of operator $Q\Gamma_{dir}Q$ has the following form

$$(3.10) \quad \Pi_{pn} = \frac{\omega^2}{2\pi^2} \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{(\tilde{q}_{|p-j|} - \tilde{q}_{p+j})(\tilde{q}_{|j-n|} - \tilde{q}_{j+n})}{j^2 - n^2}, \quad p, n \in \mathbb{N}.$$

As above, we estimate the Hilbert-Schmidt norm $\|\mathcal{V}\|_2$ of matrix $\mathcal{V} = (\Pi_{pn})$, $p, n \in \mathbb{N}$. We have:

$$\begin{aligned}
\|\mathcal{V}\|_2^2 &= \left(\frac{\omega^2}{2\pi^2} \right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{(\tilde{q}_{|p-j|} - \tilde{q}_{p+j})(\tilde{q}_{|j-n|} - \tilde{q}_{j+n})}{(j-n)(j+n)} \right|^2 \leq \\
&\leq \left(\frac{\omega^2}{\pi^2} \right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{\tilde{q}_{|p-j|}\tilde{q}_{|j-n|}}{(j-n)(j+n)} \right|^2 + \\
&+ \left(\frac{\omega^2}{\pi^2} \right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{\tilde{q}_{|p-j|}\tilde{q}_{j+n}}{(j-n)(j+n)} \right|^2 + \\
&+ \left(\frac{\omega^2}{\pi^2} \right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{\tilde{q}_{p+j}\tilde{q}_{|j-n|}}{(j-n)(j+n)} \right|^2 + \\
&+ \left(\frac{\omega^2}{\pi^2} \right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{\tilde{q}_{p+j}\tilde{q}_{j+n}}{(j-n)(j+n)} \right|^2.
\end{aligned}$$

Consider the first term in the last equality. The remaining terms are estimated in a similar way. We get

$$\begin{aligned}
\left(\frac{\omega^2}{\pi^2}\right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{\tilde{q}_{|p-j|} \tilde{q}_{|j-n|}}{(j-n)(j+n)} \right|^2 &= \left(\frac{\omega^2}{\pi^2}\right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0, l \neq -2n}} \frac{\tilde{q}_{|p-n-l|} \tilde{q}_{|l|}}{l(l+2n)} \right|^2 \\
&= \left(\frac{\omega^2}{\pi^2}\right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0, l \neq -2n}} \frac{\tilde{q}_{|l|}}{l} \left(\frac{\tilde{q}_{|p-n-l|}}{l+2n} \right) \right|^2 = \\
&= \left(\frac{\omega^2}{\pi^2}\right)^2 \sum_{p,n=1}^{\infty} \left| \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0, l \neq -2n}} \frac{\tilde{q}_{|l|}}{l} \mathcal{A}_l(p, n) \right|^2.
\end{aligned}$$

Here for $l = 0$ the matrix $\mathcal{A}_l = 0$ and for $l \neq 0$ the matrix $\mathcal{A}_l(p, n)$, $p, n \in \mathbb{N}$, has the form

$$\mathcal{A}_l(p, n) = \begin{cases} 0, & \text{if } -2n = l, \\ \frac{\tilde{q}_{|p-n-l|}}{l+2n}, & \text{if } -2n \neq l. \end{cases}$$

Every matrix \mathcal{A}_l , $l \neq 0$, is the Hilbert-Schmidt matrix and

$$\begin{aligned}
\|\mathcal{A}_l\|_2^2 &= \sum_{\substack{p,n=1 \\ -2n \neq l}}^{\infty} \left| \frac{\tilde{q}_{|p-n-l|}}{l+2n} \right|^2 \leq \sum_{\substack{n=1 \\ -2n \neq l}}^{\infty} \frac{1}{(l+2n)^2} \sum_{p=1}^{\infty} |\tilde{q}_{|p-n-l|}|^2 \leq \\
&\leq \|q\|_2^2 \sum_{\substack{n=1 \\ -2n \neq l}}^{\infty} \frac{1}{(l+2n)^2} \leq \frac{\pi^2}{6} \|q\|_2^2.
\end{aligned}$$

Using the Hölder inequality, we get

$$\begin{aligned}
\|\mathcal{V}\|_2^2 &\leq 4 \left(\frac{\omega^2}{\pi^2}\right)^2 \sum_{p,n=1}^{\infty} \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0}} \frac{|\tilde{q}_{|l|}|^2}{l^2} |\mathcal{A}_l(p, n)|^2 \leq \\
&\leq \frac{2\pi^2}{3} \|q\|_2^2 \left(\frac{\omega^2}{\pi^2}\right)^2 \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0}} \frac{|\tilde{q}_{|l|}|^2}{l^2} \leq \frac{2\omega^4}{9} \|q\|_2^4.
\end{aligned}$$

From remark 2.7 we obtain that operator $Q\Gamma_{dir}Q$ has bounded expansion on space \mathcal{H} to the Hilbert-Schmidt operator. \square

From Lemmas 3.2 – 3.4, representation (3.2) and [26, Theorem 2], we get

Theorem 3.5. *Let $l \in \mathbb{Z}_+$ for $bc \in \{per, ap\}$ (or $l \in \mathbb{N}$ for $bc = dir$) and $\|\Gamma_{l,bc}Q\|_2 \leq \frac{1}{2}$. Then the operator $L_{bc}^0 - Q$ is similar to operator $L_{bc}^0 - B$, where $B = J_{l,bc}Q + Q_0$. Here Q_0 is defined as*

$$(3.11) \quad Q_0 = Q_0(l) = (I + \Gamma_{l,bc}Q)^{-1}(Q\Gamma_{l,bc}Q - (\Gamma_{l,bc}Q)J_{l,bc}Q).$$

Also, we have

$$(L_{bc}^0 - Q)(I + \Gamma_{l,bc}Q) = (I + \Gamma_{l,bc}Q)(L_{bc}^0 - B).$$

By (3.11), the operator Q_0 has representation

$$\begin{aligned} Q_0 &= \left(\sum_{j=0}^{\infty} (-1)^j (\Gamma_{l,\theta}Q)^j \right) (Q\Gamma_{l,\theta}Q - (\Gamma_{l,\theta}Q)J_{l,\theta}Q) = \\ &= Q\Gamma_{l,\theta}Q - (\Gamma_{l,\theta}Q)J_{l,\theta}Q - \\ &\quad - (\Gamma_{l,\theta}Q) \left(\sum_{j=0}^{\infty} (-1)^j (\Gamma_{l,\theta}Q)^j \right) (Q\Gamma_{l,\theta}Q - (\Gamma_{l,\theta}Q)J_{l,\theta}Q). \end{aligned}$$

Hence, operator B has the form

$$\begin{aligned} B &= J_{l,\theta}Q + Q\Gamma_{l,\theta}Q - (\Gamma_{l,\theta}Q)J_{l,\theta}Q - \\ &\quad - (\Gamma_{l,\theta}Q) \left(\sum_{j=0}^{\infty} (-1)^j (\Gamma_{l,\theta}Q)^j \right) (Q\Gamma_{l,\theta}Q - (\Gamma_{l,\theta}Q)J_{l,\theta}Q) = \\ &= J_{l,\theta}Q + Q\Gamma_{l,\theta}Q - (\Gamma_{l,\theta}Q)J_{l,\theta}Q - \\ (3.12) \quad &\quad - (\Gamma_{l,\theta}Q)(I + \Gamma_{l,\theta}Q)^{-1}(Q\Gamma_{l,\theta}Q - (\Gamma_{l,\theta}Q)J_{l,\theta}Q). \end{aligned}$$

Theorem 3.6. *There exists a number $l \in \mathbb{Z}_+$ for $bc \in \{per, ap\}$ (or $l \in \mathbb{N}$ for $bc = dir$) and $m \geq l + 1$, such that the operator L_{bc} , $bc \in \{per, ap, dir\}$, is similar to operator $L_{bc}^0 - J_{m,bc}X_*$, where X_* is a solution of nonlinear equation*

$$(3.13) \quad X = B\Gamma_{m,bc}X - (\Gamma_{m,bc}X)(J_{m,bc}B) - (\Gamma_{m,bc}X)J_{m,bc}(B\Gamma_{m,bc}X) + B.$$

In this equation the operator B has the form (3.12) and l satisfies Theorem 3.5.

Proof. By Theorem 3.5, the operator L_{bc} is similar to the integro - differential operator $L_{bc}^0 - B$. Applying Theorem 2.14, we get the statement of this theorem. \square

Now we obtain some estimates, which will be use in the following section.

Using representations (3.12), (2.2) and (2.3) for $bc \in \{per, ap\}$ and $n \geq l + 1$, we get:

$$\begin{aligned} B\mathbb{P}_n - \mathbb{P}_n B\mathbb{P}_n &= (Q\Gamma_{l,bc}Q)\mathbb{P}_n - \mathbb{P}_n(Q\Gamma_{l,bc}Q)\mathbb{P}_n - (\Gamma_{l,bc}Q)\mathbb{P}_n Q\mathbb{P}_n + \\ &+ (\mathbb{P}_n(\Gamma_{l,bc}Q) - \Gamma_{l,bc}Q)(I + \Gamma_{l,bc}Q)^{-1}((Q\Gamma_{l,bc}Q)\mathbb{P}_n - (\Gamma_{l,bc}Q)\mathbb{P}_n Q\mathbb{P}_n). \end{aligned}$$

Note that $(J_{l,bc}Q)\mathbb{P}_n = \mathbb{P}_n Q\mathbb{P}_n$. Hence, for $n \geq l + 1$ we obtain the following estimates:

$$\begin{aligned} \|B\mathbb{P}_n - \mathbb{P}_n B\mathbb{P}_n\|_2 &\leq \|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \|\mathbb{P}_n(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \\ &+ \|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 \|\mathbb{P}_n Q\mathbb{P}_n\|_2 + \frac{\|\mathbb{P}_n(\Gamma_{l,bc}Q)\|_2 + \|\Gamma_{l,bc}Q\|_2}{1 - \|\Gamma_{l,bc}Q\|_2} \cdot \\ &\cdot (\|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 \|\mathbb{P}_n Q\mathbb{P}_n\|_2) \leq \|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \\ &+ \|\mathbb{P}_n(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 \|\mathbb{P}_n Q\mathbb{P}_n\|_2 + 2(\|\mathbb{P}_n(\Gamma_{l,bc}Q)\|_2 + \\ (3.14) \quad &+ \|\Gamma_{l,bc}Q\|_2)(\|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 \|\mathbb{P}_n Q\mathbb{P}_n\|_2). \end{aligned}$$

We will use these estimates to apply Theorem 2.17. Similar representation and inequalities hold in case $bc = dir$.

Now we estimate all term in (3.14). Arguing as in (3.4) and (3.6), we get the following estimates of norms of operators $(\Gamma_{l,bc}Q)\mathbb{P}_n$, $n \geq l + 1$, $l \in \mathbb{Z}_+$:

$$(3.15) \quad \|(\Gamma_{bc}Q)\mathbb{P}_n\|_2 \leq \frac{\omega^{2k}\|q\|_2\sqrt{2}}{4\pi^{2k}(2n + \theta)^{2k-2}(2n - 1)}, \quad k > 1, \quad \theta \in \{0, 1\},$$

and

$$(3.16) \quad \|(\Gamma_{dir}Q)P_{n,dir}\|_2 \leq \frac{\omega^{2k}\|q\|_2\sqrt{2}}{\pi^{2k}n^{2k-2}(2n - 1)}, \quad n \in \mathbb{N}, \quad k \geq 1, \quad bc = dir.$$

Now we get estimates for $\|(Q\Gamma_{bc}Q)\mathbb{P}_n\|_2$, $n \in \mathbb{Z}$, where $bc \in \{per, ap\}$. The matrix \mathfrak{A}_{bc} for operator $Q\Gamma_{bc}Q$, $bc \in \{per, ap\}$, has the following form:

$$(3.17) \quad \mathfrak{A}_{bc} = \left(\frac{\omega^{2k}}{\pi^{2k}} \left(\sum_{\substack{r \in \mathbb{Z} \\ r \neq j, r \neq -j - \theta}} \frac{q_{p-r}q_{r-j}}{(2r + \theta)^{2k} - (2j + \theta)^{2k}} \right) \right),$$

where $k > 1$, $p, r \in \mathbb{Z}$.

Then

$$\|(Q\Gamma_{bc}Q)P_n\|_2^2 = \left(\frac{\omega^{2k}}{\pi^{2k}} \right)^2 \sum_{j \in \mathbb{Z}} \left| \sum_{\substack{r \in \mathbb{Z} \\ r \neq n, r \neq -n - \theta}} \frac{q_{j-r}q_{r-n}}{(2r + \theta)^{2k} - (2n + \theta)^{2k}} \right|^2 =$$

$$\begin{aligned}
&\leq \left(\frac{\omega^{2k}}{4\pi^{2k}(2n+\theta)^{2k-2}} \right)^2 \cdot \sum_{j \in \mathbb{Z}} \left(\sum_{\substack{p \in \mathbb{Z} \\ p \neq 0}} \frac{|q_{j-p-n}|^2}{p^2} \right) \left(\sum_{\substack{p \in \mathbb{Z} \\ p \neq -2n-\theta}} \frac{|q_p|^2}{(p+2n+\theta)^2} \right) \leq \\
&\leq \left(\frac{\omega^{2k}}{4\pi^{2k}(2n+\theta)^{2k-2}} \right)^2 \sum_{\substack{p \in \mathbb{Z} \\ p \neq -2n}} \frac{|q_p|^2}{(p+2n)^2} \left(\sum_{j \in \mathbb{Z}} \sum_{\substack{p \in \mathbb{Z} \\ p \neq 0}} \frac{|q_{j-p-n}|^2}{p^2} \right) \leq \\
&\leq \left(\frac{\omega^{2k}}{4\pi^{2k}(2n+\theta)^{2k-2}} \right)^2 \frac{\pi^2}{3} \|q\|_2^2 \sum_{\substack{p \in \mathbb{Z} \\ p \neq -2n}} \frac{|q_p|^2}{(p+2n)^2} = \\
&= \left(\frac{\omega^{2k}}{4\sqrt{3}\pi^{2k-1}(2n+\theta)^{2k-2}} \right)^2 \|q\|_2^2 \sum_{\substack{r \in \mathbb{Z} \\ r \neq 0}} \frac{|q_{r-2n}|^2}{r^2} \leq \\
&\leq \left(\frac{\omega^{2k}}{4\sqrt{3}\pi^{2k-1}(2n+\theta)^{2k-2}} \right)^2 \cdot \|q\|_2^2 \left(\sum_{\substack{|r| \leq 2n \\ r \neq 0}} \frac{|q_{r-2n}|^2}{r^2} + \sum_{|r| \geq 2n+1} \frac{|q_{r-2n}|^2}{r^2} \right) \leq \\
&\leq \left(\frac{\omega^{2k}}{4\sqrt{3}\pi^{2k-1}(2n+\theta)^{2k-2}} \right)^2 \cdot \|q\|_2^2 \left(\sum_{\substack{|r| \leq 2n \\ r \neq 0}} \frac{|q_{r-2n}|^2}{r^2} + \frac{\|q\|_2^2}{(2n+\theta)^2} \right) \leq \\
&\leq \left(\frac{\omega^{2k}}{4\sqrt{3}\pi^{2k-1}(2n+\theta)^{2k-2}} \right)^2 \|q\|_2^2 \alpha^2 (2n+\theta), \quad n \in \mathbb{Z}.
\end{aligned}$$

Here α is a sequences defined in (1.1). Therefore,

$$\begin{aligned}
&\|(Q\Gamma_{bc}Q)\mathbb{P}_n\|_2^2 \leq \|(Q\Gamma_{bc}Q)P_n\|_2^2 + \|(Q\Gamma_{bc}Q)P_{-n}\|_2^2 \leq \\
(3.18) \quad &\leq 2 \left(\frac{\omega^{2k}}{4\sqrt{3}\pi^{2k-1}(2n+\theta)^{2k-2}} \right)^2 \|q\|_2^2 \alpha^2 (2n+\theta), \quad n \in \mathbb{Z}_+.
\end{aligned}$$

Further we estimate $\|(Q\Gamma_{dir}Q)P_{n,dir}\|_2$, $n \in \mathbb{N}$. For case $bc = dir$ the matrix \mathfrak{B}_{dir} of operator $Q\Gamma_{dir}Q$ has representation (3.10). Then we get

$$\|(Q\Gamma_{dir}Q)P_{n,dir}\|_2^2 =$$

$$\begin{aligned}
&= \left(\frac{\omega^{2k}}{2\pi^{2k}} \right)^2 \sum_{j=1}^{\infty} \left| \sum_{\substack{r=1 \\ r \neq n}}^{\infty} \frac{(\tilde{q}_{|j-r|} - \tilde{q}_{j+r})(\tilde{q}_{|r-n|} - \tilde{q}_{r+n})}{(r-n)(r+n)(r^{2k-2} + \dots + n^{2k-2})} \right|^2 \leq \\
&\leq \left(\frac{\omega^{2k}}{\pi^{2k}} \right)^2 \sum_{j=1}^{\infty} \left| \sum_{\substack{r=1 \\ r \neq n}}^{\infty} \frac{\tilde{q}_{|j-r|}\tilde{q}_{|r-n|}}{(r-n)(r+n)(r^{2k-2} + \dots + n^{2k-2})} \right|^2 + \\
&+ \left(\frac{\omega^{2k}}{\pi^{2k}} \right)^2 \sum_{j=1}^{\infty} \left| \sum_{\substack{r=1 \\ r \neq n}}^{\infty} \frac{\tilde{q}_{|j-r|}\tilde{q}_{r+n}}{(r-n)(r+n)(r^{2k-2} + \dots + n^{2k-2})} \right|^2 + \\
&+ \left(\frac{\omega^{2k}}{\pi^{2k}} \right)^2 \sum_{j=1}^{\infty} \left| \sum_{\substack{r=1 \\ r \neq n}}^{\infty} \frac{\tilde{q}_{j+r}\tilde{q}_{|r-n|}}{(r-n)(r+n)(r^{2k-2} + \dots + n^{2k-2})} \right|^2 + \\
&+ \left(\frac{\omega^{2k}}{\pi^{2k}} \right)^2 \sum_{j=1}^{\infty} \left| \sum_{\substack{r=1 \\ r \neq n}}^{\infty} \frac{\tilde{q}_{j+r}\tilde{q}_{r+n}}{(r-n)(r+n)(r^{2k-2} + \dots + n^{2k-2})} \right|^2.
\end{aligned}$$

Consider the first term in the last inequality. The remaining terms are estimated in a similar way by the same constant. We obtain

$$\begin{aligned}
&\left(\frac{\omega^{2k}}{\pi^{2k}} \right)^2 \sum_{j=1}^{\infty} \left| \sum_{\substack{r=1 \\ r \neq n}}^{\infty} \frac{\tilde{q}_{|j-r|}\tilde{q}_{|r-n|}}{(r-n)(r+n)(r^{2k-2} + \dots + n^{2k-2})} \right|^2 = \\
&= \left(\frac{\omega^{2k}}{\pi^{2k}} \right)^2 \sum_{j=1}^{\infty} \left| \sum_{\substack{p \in \mathbb{Z} \\ p \neq 0, p \neq -2n}} \frac{\tilde{q}_{|j-p-n|}\tilde{q}_{|p|}}{p(p+2n)((n+p)^{2k-2} + \dots + n^{2k-2})} \right|^2 \leq \\
&\leq \left(\frac{\omega^{2k}}{\pi^{2k}n^{2k-2}} \right)^2 \sum_{j=1}^{\infty} \left(\sum_{\substack{p \in \mathbb{Z} \\ p \neq 0}} \frac{|\tilde{q}_{|j-p-n|}|^2}{p^2} \right) \left(\sum_{\substack{p \in \mathbb{Z} \\ p \neq -2n}} \frac{|\tilde{q}_{|p|}|^2}{(p+2n)^2} \right) \leq \\
&\leq \left(\frac{\omega^{2k}}{\pi^{2k}n^{2k-2}} \right)^2 \frac{\pi^2}{3} \|q\|_2^2 \sum_{\substack{p \in \mathbb{Z} \\ p \neq -2n}} \frac{|\tilde{q}_{|p|}|^2}{(p+2n)^2} = \\
&= \left(\frac{\omega^{2k}}{\pi^{2k-1}n^{2k-2}\sqrt{3}} \right)^2 \|q\|_2^2 \sum_{\substack{r \in \mathbb{Z} \\ r \neq 0}} \frac{|\tilde{q}_{|r-2n|}|^2}{r^2} \leq \\
&\leq \left(\frac{\omega^{2k}}{\pi^{2k-1}n^{2k-2}\sqrt{3}} \right)^2 \|q\|_2^2 \left(\sum_{\substack{|r| \leq 2n \\ r \neq 0}} \frac{|\tilde{q}_{|r-2n|}|^2}{r^2} + \sum_{|r| \geq 2n+1} \frac{|\tilde{q}_{|r-2n|}|^2}{r^2} \right) \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{\omega^{2k}}{\pi^{2k-1} n^{2k-2} \sqrt{3}} \right)^2 \|q\|_2^2 \left(\sum_{\substack{|r| \leq 2n \\ r \neq 0}} \frac{|\tilde{q}_{|r-2n|}|^2}{r^2} + \frac{\|q\|_2^2}{(2n+1)^2} \right) \leq \\
&\leq \left(\frac{\omega^{2k}}{\pi^{2k-1} n^{2k-2} \sqrt{3}} \right)^2 \|q\|_2^2 \beta^2(2n).
\end{aligned}$$

Finally, we obtain

$$(3.19) \quad \|(Q\Gamma_{dir}Q)P_{n,dir}\|_2 \leq \frac{2\omega^{2k}}{\pi^{2k-1} n^{2k-2} \sqrt{3}} \|q\|_2 \beta(2n), \quad n \geq m+1,$$

where β is a sequence defined in (1.2).

In case $k = 1$ we have analogous inequality:

$$(3.20) \quad \|(Q\Gamma_{dir}Q)P_{n,dir}\|_2 \leq \frac{2\omega^2}{\pi\sqrt{3}} \|q\|_2 \beta(2n), \quad n \geq m+1.$$

Finally, consider $\|P_{n,dir}(Q\Gamma_{dir}Q)P_{n,dir}\|_2$, $n \in \mathbb{N}$, for case $k = 1$. We have the following inequalities:

$$\begin{aligned}
&\|P_{n,dir}(Q\Gamma_{dir}Q)P_{n,dir}\|_2 = |(Q\Gamma_{dir}Qe_{n,dir}, e_{n,dir})| = \\
&= \frac{\omega^2}{2\pi^2} \left| \sum_{\substack{l=1 \\ l \neq n}}^{\infty} \frac{(\tilde{q}_{|n-l|} - \tilde{q}_{n+l})(\tilde{q}_{|l-n|} - \tilde{q}_{l+n})}{(l-n)(l+n)} \right| = \\
&= \frac{\omega^2}{2\pi^2} \left| \sum_{\substack{p \in \mathbb{Z} \\ p \neq 0, p \neq -2n}} \frac{(\tilde{q}_{|p|} - \tilde{q}_{|p+2n|})^2}{p(p+2n)} \right| = \\
&= \frac{\omega^2}{4\pi^2 n} \left| \sum_{\substack{p \in \mathbb{Z} \\ p \neq 0}} \frac{(\tilde{q}_{|p|} - \tilde{q}_{|p+2n|})^2}{p} - \sum_{\substack{p \in \mathbb{Z} \\ p \neq -2n}} \frac{(\tilde{q}_{|p|} - \tilde{q}_{|p+2n|})^2}{p+2n} \right| \leq \frac{\omega^2}{2\pi^2 n} \left(\frac{|\tilde{q}_{2n}|^2}{2n} + \right. \\
&+ \left| \sum_{\substack{p \in \mathbb{Z} \\ p \neq 0}} \frac{|\tilde{q}_{|p|}|^2 + |\tilde{q}_{|p+2n|}|^2}{p} \right| + \left| \sum_{\substack{p \in \mathbb{Z} \\ p \neq -2n}} \frac{|\tilde{q}_{|p|}|^2 + |\tilde{q}_{|p+2n|}|^2}{p+2n} \right| \Big) \leq \\
&\leq \frac{\omega^2}{2\pi^2 n} \left(\frac{\|q\|_2^2}{2n} + \frac{2\|q\|_2^2 \pi^2}{3} + \|q\|_2 \left(\sum_{\substack{p \in \mathbb{Z} \\ p \neq 0}} \frac{|\tilde{q}_{|p+2n|}|^2}{p^2} \right)^{\frac{1}{2}} + \right. \\
&+ \left. \|q\|_2 \left(\sum_{\substack{j \in \mathbb{Z} \\ j \neq 0}} \frac{|\tilde{q}_{|j-2n|}|^2}{j^2} \right)^{\frac{1}{2}} \right) \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\omega^2}{2\pi^2 n} \left(\frac{\|q\|_2^2}{2n} + \frac{2\|q\|_2^2 \pi^2}{3} + \|q\|_2 \left(\sum_{\substack{|p| \leq 2n \\ p \neq 0}} \frac{|\tilde{q}_{|p+2n|}|^2}{p^2} + \frac{\|q\|_2^2}{(2n+1)^2} \right)^{\frac{1}{2}} + \right. \\
&\quad \left. + \|q\|_2 \left(\sum_{\substack{|j| \leq 2n \\ j \neq 0}} \frac{|\tilde{q}_{|j-2n|}|^2}{j^2} + \frac{\|q\|_2^2}{(2n+1)^2} \right)^{\frac{1}{2}} \right) \leq \frac{\tilde{C}}{n} \beta(2n),
\end{aligned}$$

where \tilde{C} is some constant. Therefore,

$$(3.21) \quad \|P_{n,dir}(Q\Gamma_{dir}Q)P_{n,dir}\|_2 \leq \frac{\tilde{C}}{n} \beta(2n).$$

Remark 3.7. We get analogous estimates for $\|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2$, $\|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2$, $n \geq l+1$, $bc \in \{per, ap\}$. This follows from definition of operators $\Gamma_{l,bc}Q$. This statement also holds for case $bc = dir$.

4. PROOF OF THE MAIN RESULTS

The proof of the main results is based on preliminary transformation of similarity and estimates from Section 3.

Proof of Theorem 1.1. We transform operator L_{bc} , $bc \in \{per, ap\}$, to operator $L_{bc}^0 - B$, where B has form (3.12). Since B belongs to $\mathfrak{S}_2(\mathcal{H})$, then we can may apply results of Section 2 to operator $L_{bc}^0 - B$. By Theorem 3.6, there exists a number $m \in \mathbb{Z}_+$, $m \geq l+1$ (see condition (2.9)), such that operator L_{bc} is similar to operator $L_{bc}^0 - J_{m,bc}X_*$, where X_* is a solution of equation (3.13). Since operator $J_{m,bc}X_*$ is bounded and L_{bc}^0 is self-adjoint operator with compact resolvent, then $L_{bc}^0 - J_{m,bc}X_*$ is operator with compact resolvent. Hence, the similar operator L_{bc} is operator with compact resolvent.

We consider the sequence A_n , $n \geq m+1$, of a restriction of operator $\mathbb{P}_n X_* \mathbb{P}_n$ on subspace $\text{Im } \mathbb{P}_n$. By $A_{(m)}$ we denote a restriction of operator $L_{bc}^0 - \mathbb{P}_{(m)} X_* \mathbb{P}_{(m)}$ on subspace $\text{Im } \mathbb{P}_{(m)}$. The similarity of operators L_{bc} and $L_{bc}^0 - J_{m,bc}X_*$ provides that $\sigma(L_{bc}^0 - J_{m,bc}X_*) = \sigma(L_{bc})$. Therefore,

$$(4.1) \quad \sigma(L_{bc}) = \sigma_{(m)} \bigcup \left(\bigcup_{n \geq m+1} \left(\left(\frac{\pi(2n+\theta)}{\omega} \right)^{2k} - \sigma_n \right) \right), \quad k > 1,$$

where $\sigma_{(m)} = \sigma(A_{(m)})$ and $\sigma_n = \sigma(A_n)$. Note that sets $\sigma_{(m)}$ and σ_n are mutually disjoint.

To obtain the asymptotic of eigenvalues for operator L_{bc} we need to describe sets σ_n , $n \geq m+1$. The matrix \mathcal{A}_n of operator A_n , $n \geq m+1$,

in the basis $e_{-n-\theta}, e_n$ of subspace $\text{Im } \mathbb{P}_n$ has the form

$$\mathcal{A}_n = \begin{pmatrix} (X_* e_{-n-\theta}, e_{-n-\theta}) & (X_* e_n, e_{-n-\theta}) \\ (X_* e_{-n-\theta}, e_n) & (X_* e_n, e_n) \end{pmatrix}.$$

Operators $A_n, n \geq m+1$, are represented as

$$A_n = B_n + C_n + D_n, \quad n \geq m+1,$$

where B_n, C_n and D_n are restrictions of operators $\mathbb{P}_n(J_{m,bc}Q)\mathbb{P}_n = \mathbb{P}_n Q \mathbb{P}_n, \mathbb{P}_n(B - J_{m,bc}Q)\mathbb{P}_n$ and $\mathbb{P}_n(X_* - B)\mathbb{P}_n$ on subspace \mathcal{H}_n , respectively. Then we have

$$(4.2) \quad \mathcal{A}_n = \mathcal{B}_n + \mathcal{C}_n + \mathcal{D}_n,$$

where

$$\mathcal{B}_n = \begin{pmatrix} 0 & q_{-2n-\theta} \\ q_{2n+\theta} & 0 \end{pmatrix},$$

$$\mathcal{C}_n = \begin{pmatrix} (Q\Gamma_{bc}Qe_{-n-\theta}, e_{-n-\theta}) & (Q\Gamma_{bc}Qe_n, e_{-n-\theta}) \\ (Q\Gamma_{bc}Qe_{-n-\theta}, e_n) & (Q\Gamma_{bc}Qe_n, e_n) \end{pmatrix},$$

and \mathcal{D}_n is the matrix of operator D_n .

Applying formulas (3.12), (3.13) for $X = X_*$ and Theorem 2.17, we get

$$(4.3) \quad \begin{aligned} & \|\mathbb{P}_n(X_* - B)\mathbb{P}_n\|_2 \leq \\ & \leq \frac{\omega^{2k}}{2\pi^{2k}(2n-1)(2n+\theta)^{2k-2}} \|\mathbb{P}_n B - \mathbb{P}_n B \mathbb{P}_n\|_2 \|B \mathbb{P}_n - \mathbb{P}_n B \mathbb{P}_n\|_2 \leq \\ & \leq \frac{\omega^{2k}}{2\pi^{2k}(2n-1)(2n+\theta)^{2k-2}} \|B\|_2 \|B \mathbb{P}_n - \mathbb{P}_n B \mathbb{P}_n\|_2, \quad k > 1, n \geq n_0. \end{aligned}$$

Here $n_0 = \max\{m+1, \frac{1}{2}(\frac{3\|B\|_2\omega^{2k}}{\pi^{2k}})^{\frac{1}{2k-1}}\}$.

Using representations (3.14), formulas (3.2), (3.4), (3.15), (3.18) and remark 3.7, we have the following inequalities:

$$\begin{aligned} & \|B \mathbb{P}_n - \mathbb{P}_n B \mathbb{P}_n\|_2 \leq \|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \|\mathbb{P}_n(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \\ & + \|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 \|\mathbb{P}_n Q \mathbb{P}_n\|_2 + 2(\|\mathbb{P}_n(\Gamma_{l,bc}Q)\|_2 + \\ & + \|\Gamma_{l,bc}Q\|_2)(\|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 \|\mathbb{P}_n Q \mathbb{P}_n\|_2) \leq \\ & \leq 2\|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 \|\mathbb{P}_n Q \mathbb{P}_n\|_2 + \\ & + 2(\|\mathbb{P}_n(\Gamma_{l,bc}Q)\|_2 + \|\Gamma_{l,bc}Q\|_2)(\|(Q\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 + \\ & + \|(\Gamma_{l,bc}Q)\mathbb{P}_n\|_2 \|\mathbb{P}_n Q \mathbb{P}_n\|_2) \leq \\ & \leq \frac{\omega^{2k}\|q\|_2}{\pi^{2k-1}(2n+\theta)^{2k-2}\sqrt{6}} \alpha(2n+\theta) + \frac{\sqrt{2}\omega^{2k}\|q\|_2^2}{4\pi^{2k}(2n+\theta)^{2k-2}(2n-1)} + \end{aligned}$$

$$(4.4) \quad + 2 \left(\frac{\sqrt{2}\omega^{2k}\|q\|_2}{4\pi^{2k}(2n+\theta)^{2k-2}(2n-1)} + 1 \right) \left(\frac{\omega^{2k}\|q\|_2\alpha(2n+\theta)}{2\sqrt{6}\pi^{2k-1}(2n+\theta)^{2k-2}} + \frac{\sqrt{2}\omega^{2k}\|q\|_2^2}{4\pi^{2k}(2n+\theta)^{2k-2}(2n-1)} \right) \leq \frac{C_3\alpha(2n+\theta)}{(2n+\theta)^{2k-2}}, \quad n \geq m+1.$$

Here $C_3 > 0$ is some constant.

By inequalities (4.3) and (4.4), we obtain the estimates for sequences of matrices \mathcal{D}_n , $n \geq m+1$:

$$(4.5) \quad \|\mathcal{D}_n\|_2 \leq \frac{C_4}{n(2n+\theta)^{4k-4}}\alpha(2n+\theta), \quad n \geq m+1,$$

where $C_4 > 0$ is some constant.

Let us start to estimate the eigenvalues of operators A_n , $n \geq m+1$. First of all, we compute the elements of matrix \mathcal{C}_n . We have

$$(4.6) \quad c_{11}(n) = (Q\Gamma_{bc}Qe_{-n-\theta}, e_{-n-\theta}) = \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \neq n \\ j \neq -n-\theta}} \frac{q_{-n-j-\theta}q_{n+j+\theta}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}},$$

$$(4.7) \quad c_{12}(n) = (Q\Gamma_{bc}Qe_n, e_{-n-\theta}) = \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \neq n \\ j \neq -n-\theta}} \frac{q_{-n-j-\theta}q_{j-n}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}},$$

$$(4.8) \quad c_{21}(n) = (Q\Gamma_{bc}Qe_{-n-\theta}, e_n) = \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \neq n \\ j \neq -n-\theta}} \frac{q_{n-j}q_{n+j+\theta}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}},$$

$$(4.9) \quad c_{22}(n) = (Q\Gamma_{bc}Qe_n, e_n) = \frac{\omega^{2k}}{\pi^{2k}} \sum_{\substack{j \neq n \\ j \neq -n-\theta}} \frac{q_{n-j}q_{j-n}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}}.$$

Obviously, $c_{11}(n) = c_{22}(n)$.

Note that for all sequences of complex numbers a_n , b_n , $n \geq 1$, satisfying $a_nb_n \neq 0$, we have

$$\begin{pmatrix} -\sqrt{a_nb_n} & 0 \\ 0 & \sqrt{a_nb_n} \end{pmatrix} = U_n \begin{pmatrix} 0 & a_n \\ b_n & 0 \end{pmatrix} U_n^{-1}, \quad n \geq 1,$$

where $U_n = \begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{b_n}{a_n}} & \sqrt{\frac{b_n}{a_n}} \end{pmatrix}$ and $U_n^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{a_n}{b_n}} \\ \frac{1}{2} & \frac{1}{2}\sqrt{\frac{a_n}{b_n}} \end{pmatrix}$.

Put $a_n = q_{-2n-\theta} + c_{12}(n)$ and $b_n = q_{2n+\theta} + c_{21}(n)$. Multiply the matrix $\mathcal{B}_n + \mathcal{C}_n$ by U_n from the left and by U_n^{-1} from the right, we get

that the matrix $\mathcal{B}_n + \mathcal{C}_n + \mathcal{D}_n$ is similar to

$$\begin{pmatrix} \mu_n^- & 0 \\ 0 & \mu_n^+ \end{pmatrix} + U_n \mathcal{D}_n U_n^{-1} = \Lambda_n + \tilde{D}_n, \quad n \geq m+1,$$

where

$$\begin{aligned} \mu_n^- &= -\sqrt{(q_{-2n-\theta} + c_{12}(n))(q_{2n+\theta} + c_{21}(n))}, \\ \mu_n^+ &= \sqrt{(q_{-2n-\theta} + c_{12}(n))(q_{2n+\theta} + c_{21}(n))}, \end{aligned}$$

and $\tilde{D}_n = U_n \mathcal{D}_n U_n^{-1}$. Since Λ_n is diagonal matrix, then eigenvalues $\tilde{\mu}_n^\mp$ of matrix $\Lambda_n + \tilde{D}_n$, $n \geq m+1$, (the spectrum of this matrix coincides with σ_n) satisfy the estimates

$$\begin{aligned} |\tilde{\mu}_n^\mp - \mu_n^\mp| &\leq \|\tilde{D}_n\|_2 \leq \|U_n\|_2 \|U_n^{-1}\|_2 \|\mathcal{D}_n\|_2 \leq \\ &\leq \frac{C_4 w_\theta(n)}{n(2n+\theta)^{4k-4}} \alpha(2n+\theta) = \frac{C_\theta}{n^{4k-3}} w_\theta(n) \alpha(2n+\theta), \quad n \geq m+1. \end{aligned}$$

Here we use the estimate (4.5).

Finally, we estimate $c_{11}(n)$. Arguing as (3.18), we get

$$\begin{aligned} |c_{11}(n)| &= |(Q\Gamma_{bc}Qe_{-n-\theta}, e_{-n-\theta})| = \\ &= \frac{\omega^{2k}}{\pi^{2k}} \left| \sum_{\substack{j \neq n \\ j \neq -n-\theta}} \frac{q_{-n-j} q_{n+j}}{(2j+\theta)^{2k} - (2n+\theta)^{2k}} \right| \leq \frac{\omega^{2k} \|q\|_2 \alpha(2n+\theta)}{2\sqrt{6}\pi^{2k-1} (2n+\theta)^{2k-2}}. \end{aligned}$$

Hence, the eigenvalues $\tilde{\lambda}_n^\mp$ of the operator L_{bc} satisfy the following estimates

$$\begin{aligned} &\left| \tilde{\lambda}_n^\mp - \left(\frac{\pi(2n+\theta)}{\omega} \right)^{2k} + q_0 + c_{11}(n) \pm \right. \\ (4.10) \quad &\left. \pm \sqrt{(q_{-2n-\theta} + c_{12}(n))(q_{2n+\theta} + c_{21}(n))} \right| \leq \frac{C_\theta}{n^{4k-3}} w_\theta(n) \alpha(2n+\theta). \end{aligned}$$

for $n \geq m+1$. The Theorem is proved.

Proof of Theorem 1.2. If q is a function of bounded variation, then for the Fourier coefficients q_n , $n \in \mathbb{Z}$, we have

$$|q_n| \leq \frac{C_5}{|n|+1}, \quad n \in \mathbb{Z}.$$

Here $C_5 > 0$ is some constant. These estimates can be found in [31, Theorem II.4.12]. Using formula (1.1), we get

$$\alpha(2n+\theta) \leq \frac{\tilde{C}_5}{|n|+1}, \quad n \in \mathbb{Z}_+,$$

where $\tilde{C}_5 > 0$ is some constant. Applying Theorem 1.1, we obtain the statement of Theorem 1.2. The Theorem is proved.

Corollary 1.3 follows from Theorem 1.2.

In comparison with [15, Theorem 3.1], Corollary 1.3 improves the asymptotic representation and the remainder term of eigenvalues $\tilde{\lambda}_n^\mp$.

Remark 4.1. There is an error in remainder term in the original version of this result (see [15, Proof of Theorem 3.1, page 5]).

Proof of Theorem 1.4. We transform operator L_{dir} to operator $L_{dir}^0 - B$, where B has the form (3.12). By Theorem 3.6 the operator $L_{dir}^0 - B$ is similar to $L_{dir}^0 - J_{m,dir}X_*$, where X_* is a solution of equation (3.13). Hence, the operator L_{dir} has compact resolvent.

Since projections $P_{n,dir}$, $n \geq m+1$, have rang one, then operators $P_{n,dir}X_*P_{n,dir}$ have the following form $P_{n,dir}X_*P_{n,dir} = (X_*e_{n,dir}, e_{n,dir})P_{n,dir}$, $n \geq m+1$. Therefore,

$$\sigma(L_{dir}) = \sigma_{(m)} \cup \left(\bigcup_{n \geq m+1} \left\{ \left(\frac{\pi n}{\omega} \right)^{2k} - (X_*e_{n,dir}, e_{n,dir}) \right\} \right), \quad k > 1,$$

where $\sigma_{(m)}$ is a restriction of operator $L_{dir}^0 - J_{m,dir}X_*$ on subspace $\text{Im } P_{(m)}$.

Sequence $(X_*e_{n,dir}, e_{n,dir})$ has the form

$$\begin{aligned} (X_*e_{n,dir}, e_{n,dir}) &= (Be_{n,dir}, e_{n,dir}) + ((X_* - B)e_{n,dir}, e_{n,dir}) = \\ &= (P_{n,dir}QP_{n,dir}e_{n,dir}, e_{n,dir}) + (Q\Gamma_{l,dir}Qe_{n,dir}, e_{n,dir}) + \eta_{dir}(n) = \\ &= \frac{1}{\omega} \int_0^\omega q(t) dt - \frac{1}{\omega} \int_0^\omega q(t) \cos \frac{2\pi n}{\omega} t dt + \\ &+ (Q\Gamma_{l,dir}Qe_{n,dir}, e_{n,dir}) + \eta_{dir}(n). \end{aligned}$$

Next,

$$(Q\Gamma_{l,dir}Qe_{n,dir}, e_{n,dir}) = \frac{\omega^{2k}}{2\pi^{2k}} \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{(\tilde{q}_{|n-j|} - \tilde{q}_{n+j})^2}{j^{2k} - n^{2k}}.$$

Now we estimate $|\eta_{dir}(n)|$, $n \geq m+1$. Using inequality (2.18) of Theorem 2.18, we get

$$\begin{aligned} |\eta_{dir}(n)| &= |((X_* - B)e_{n,dir}, e_{n,dir})| \leq \|P_{n,dir}(X_* - B)P_{n,dir}\|_2 \leq \\ &\leq \frac{2\omega^{2k}}{\pi^{2k}(2n-1)n^{2k-2}} \cdot \|P_{n,dir}B - P_{n,dir}BP_{n,dir}\|_2 \|BP_{n,dir} - P_{n,dir}BP_{n,dir}\|_2 \leq \\ &\leq \frac{2\omega^{2k}\|B\|_2}{\pi^{2k}(2n-1)n^{2k-2}} \|BP_{n,dir} - P_{n,dir}BP_{n,dir}\|_2, \quad n \geq n_1. \end{aligned}$$

Here $n_1 = \{m + 1, (\frac{6\|B\|_2\omega^{2k}}{\pi^{2k}})^{\frac{1}{2k-1}}\}$.

Finally, we estimate the value $\|BP_{n,dir} - P_{n,dir}BP_{n,dir}\|_2$. Combining (3.2), (3.6), (3.16), (3.19), (3.21) and representation (3.14) for $P_{n,dir}$, we obtain

$$\begin{aligned}
& \|BP_{n,dir} - P_{n,dir}BP_{n,dir}\|_2 \leq \|(Q\Gamma_{l,dir}Q)P_{n,dir}\|_2 + \\
& + \|P_{n,dir}(Q\Gamma_{l,dir}Q)P_{n,dir}\|_2 + \|(\Gamma_{l,dir}Q)P_{n,dir}\|_2 \|J_{l,dir}Q\|_2 + \\
& + 2(\|P_{n,dir}(\Gamma_{l,dir}Q)\|_2 + \|\Gamma_{l,dir}Q\|_2)(\|(Q\Gamma_{l,dir}Q)P_{n,dir}\|_2 + \\
& + \|(\Gamma_{l,dir}Q)P_{n,dir}\|_2 \|J_{l,dir}Q\|_2) \leq \\
& \leq 2\|(Q\Gamma_{l,dir}Q)P_{n,dir}\|_2 + \|(\Gamma_{l,dir}Q)P_{n,dir}\|_2 \|J_{l,dir}Q\|_2 + \\
& + 2(\|P_{n,dir}(\Gamma_{l,dir}Q)\|_2 + \|\Gamma_{l,dir}Q\|_2)(\|(Q\Gamma_{l,dir}Q)P_{n,dir}\|_2 + \\
& + \|(\Gamma_{l,dir}Q)P_{n,dir}\|_2 \|J_{l,dir}Q\|_2) \leq \frac{4\omega^{2k}\|q\|_2}{\pi^{2k-1}n^{2k-2}\sqrt{3}}\beta(2n) + \\
& + \frac{\omega^{2k}\|q\|_2^2\sqrt{2}}{\pi^{2k}n^{2k-2}(2n-1)} + 2\left(\frac{\omega^{2k}\|q\|_2\sqrt{2}}{\pi^{2k}n^{2k-2}(2n-1)} + 1\right) \cdot \\
& \cdot \left(\frac{2\omega^{2k}\|q\|_2}{\pi^{2k-1}n^{2k-2}\sqrt{3}}\beta(2n) + \frac{\omega^{2k}\|q\|_2^2\sqrt{2}}{\pi^{2k}n^{2k-2}(2n-1)}\right) \leq \frac{C_6}{n^{2k-2}}\beta(2n), \quad n \geq n_1.
\end{aligned}$$

Here $C_6 > 0$ is some constant.

Hence, the sequence $\eta_{dir}(n)$, $n \geq n_1$, has the estimate

$$|\eta_{dir}(n)| \leq \frac{M}{n^{4k-3}}\beta(2n), \quad n \geq n_1,$$

where $M > 0$ is some constant. Hence, we get asymptotic representation (1.7).

If q is a function of bounded variation, we apply the scheme of the proof of Theorem 1.2 and we get

$$\begin{aligned}
\tilde{\lambda}_{n,dir} &= \left(\frac{\pi n}{\omega}\right)^{2k} - \frac{1}{\omega} \int_0^\omega q(t) dt + \frac{1}{\omega} \int_0^\omega q(t) \cos \frac{2\pi n}{\omega} t dt - \\
&- \frac{\omega^{2k}}{2\pi^{2k}} \sum_{\substack{j=1 \\ j \neq n}}^\infty \frac{(\tilde{q}_{|n-j|} - \tilde{q}_{n+j})^2}{j^{2k} - n^{2k}} + \mathcal{O}\left(\frac{1}{n^{4k-2}}\right), \quad n \geq m+1, \quad k > 1.
\end{aligned}$$

The Theorem is proved.

Theorem 1.4 is stronger than the corresponding result from [14, Theorem 1].

Proof of Theorem 1.5. Suppose $k = 1$. As in the proof of Theorem 1.4, we get that the operator L_{dir} is operator with compact resolvent.

The spectrum of operator L_{dir} has the following form

$$\sigma(L_{dir}) = \sigma_{(m)} \bigcup \left(\bigcup_{n \geq m+1} \left\{ \left(\frac{\pi n}{\omega} \right)^2 - (X_* e_{n,dir}, e_{n,dir}) \right\} \right),$$

where $\sigma_{(m)}$ is a restriction of operator $L_{dir}^0 - J_{m,dir} X_*$ on subspace $\text{Im } P_{(m)}$.

Next,

$$\begin{aligned} (X_* e_{n,dir}, e_{n,dir}) &= (B e_{n,dir}, e_{n,dir}) + ((X_* - B) e_{n,dir}, e_{n,dir}) = \\ &= (P_{n,dir} Q P_{n,dir} e_{n,dir}, e_{n,dir}) + (Q \Gamma_{l,dir} Q e_{n,dir}, e_{n,dir}) + \eta_{dir}(n) = \\ &= \frac{1}{\omega} \int_0^\omega q(t) dt - \frac{1}{\omega} \int_0^\omega q(t) \cos \frac{2\pi n}{\omega} t dt + (Q \Gamma_{l,dir} Q e_{n,dir}, e_{n,dir}) + \zeta(n). \end{aligned}$$

Now we estimate $|\zeta(n)|$, $n \geq m+1$. Using inequality (2.18) of Theorem 2.18, we get

$$\begin{aligned} |\zeta(n)| &= |((X_* - B) e_{n,dir}, e_{n,dir})| \leq \|P_{n,dir} (X_* - B) P_{n,dir}\|_2 \leq \\ &\leq \frac{2\omega^2}{\pi^2(2n-1)} \|P_{n,dir} B - P_{n,dir} B P_{n,dir}\|_2 \|B P_{n,dir} - P_{n,dir} B P_{n,dir}\|_2 \leq \\ &\leq \frac{2\omega^2 \|B\|_2}{\pi^2(2n-1)} \|B P_{n,dir} - P_{n,dir} B P_{n,dir}\|_2, \quad n \geq n_1. \end{aligned}$$

Here $n_1 = \{m+1, \frac{6\|B\|_2\omega^2}{\pi^2} + \frac{1}{2}\}$.

Finally, we estimate the value $\|B P_{n,dir} - P_{n,dir} B P_{n,dir}\|_2$. Combining (3.2), (3.6), (3.20), (3.21) and representation (3.14) for $P_{n,dir}$, we obtain

$$\begin{aligned} \|B P_{n,dir} - P_{n,dir} B P_{n,dir}\|_2 &\leq \|(Q \Gamma_{l,dir} Q) P_{n,dir}\|_2 + \\ &+ \|P_{n,dir} (Q \Gamma_{l,dir} Q) P_{n,dir}\|_2 + \|(\Gamma_{l,dir} Q) P_{n,dir}\|_2 \|J_{l,dir} Q\|_2 + \\ &+ 2(\|P_{n,dir} (\Gamma_{l,dir} Q)\|_2 + \|\Gamma_{l,dir} Q\|_2) (\|(Q \Gamma_{l,dir} Q) P_{n,dir}\|_2 + \\ &+ \|(\Gamma_{l,dir} Q) P_{n,dir}\|_2 \|J_{l,dir} Q\|_2) \leq \frac{2\omega^2 \|q\|_2}{\pi\sqrt{3}} \beta(2n) + \frac{\omega^2 \|q\|_2 \sqrt{2}}{\pi^2 n} \beta(2n) + \\ &+ 2 \left(\frac{\omega^2 \|q\|_2^2 \sqrt{2}}{\pi^2(2n-1)} + 1 \right) \left(\frac{2\omega^2 \|q\|_2}{\pi\sqrt{3}} \beta(2n) + \frac{\omega^2 \|q\|_2^2 \sqrt{2}}{\pi^2(2n-1)} \right) \leq C_7 \beta(2n) \end{aligned}$$

for $n \geq n_1$. Here $C_7 > 0$ is some constant.

Hence, the sequence $\zeta(n)$, $n \geq n_1$, satisfies the inequality

$$|\zeta(n)| \leq \frac{\widetilde{M}}{n} \beta(2n), \quad n \geq n_1,$$

where $\widetilde{M} > 0$ is some constant. Considering inequality (3.21), we obtain the asymptotic representation (1.7).

If q is a function of bounded variation, then we apply the scheme of the proof of theorem 1.2 and for $n \geq m + 1$ we get:

$$\tilde{\lambda}_{n,dir} = \left(\frac{\pi n}{\omega}\right)^2 - \frac{1}{\omega} \int_0^\omega q(t) dt + \frac{1}{\omega} \int_0^\omega q(t) \cos \frac{2\pi n}{\omega} t dt + \mathcal{O}\left(\frac{1}{n^2}\right).$$

The Theorem is proved.

The results of Theorem 1.5 are more exact than earlier results [14, Theorem 1], [19, Theorem 0.3], [20, Theorem 1.2], [22, Theorem 1].

Proof of Corollary 1.6. By Theorem 3.6, operator L_{dir} is similar to

$$L_{dir}^0 - J_{m,dir} X_* = L_{dir}^0 - P_{(m)} X_* P_{(m)} - \sum_{n \geq m+1} P_{n,dir} X_* P_{n,dir}.$$

The restriction of this operator on subspace $\text{Ker } P_{(m)} = \text{Im } (I - P_{(m)})$ is a scalar type operator. Hence, L_{dir} is spectral (in the Dunford sense). The Corollary is proved.

Proof of theorem 1.8. Suppose $bc \in \{per, ap\}$. By Theorem 3.5, the operator L_{bc} is similar to operator $L_{bc}^0 - B$ and

$$(L_{bc}^0 - Q)(I + \Gamma_{l,bc} Q) = (I + \Gamma_{l,bc} Q)(L_{bc}^0 - B),$$

where B is defined by formula (3.12). Taking into account theorem 3.6, we conclude there exists a number $m \in \mathbb{Z}_+$, $m \geq l + 1$ (see condition (2.9)), such that the operator L_{bc} is similar to $L_{bc}^0 - J_{m,bc} X_*$, where X_* is a solution of equation (3.13). Then

$$(L_{bc}^0 - B)(I + \Gamma_{m,bc} X_*) = (I + \Gamma_{m,bc} X_*)(L_{bc}^0 - J_{m,bc} X_*).$$

Using these equalities, we get

$$\begin{aligned} L_{bc}^0 - Q &= (I + \Gamma_{l,bc} Q)(I + \Gamma_{m,bc} X_*) \cdot \\ &\cdot (L_{bc}^0 - J_{m,bc} X_*)(I + \Gamma_{m,bc} X_*)^{-1} (I + \Gamma_{l,bc} Q)^{-1}. \end{aligned}$$

Hence, from [26, Lemma 1] for projections defined in introduction we obtain

$$\begin{aligned} P(\tilde{\Delta}, L_{bc}) &= (I + \Gamma_{l,bc} Q)(I + \Gamma_{m,bc} X_*) P(\Delta, L_{bc}^0) \cdot \\ (4.11) \quad &\cdot (I + \Gamma_{m,bc} X_*)^{-1} (I + \Gamma_{l,bc} Q)^{-1}. \end{aligned}$$

Here $\Delta = \Delta(\Omega) = \cup_{n \in \Omega} \{\lambda_n\}$, $\tilde{\Delta} = \tilde{\Delta}(\Omega) = \cup_{n \in \Omega} \sigma_n$ and σ_n is defined in (4.1), $\Omega \subset \{m+1, m+2, \dots\}$. Using (4.11), we have

$$\begin{aligned}
P(\tilde{\Delta}, L_{bc}) - P(\Delta, L_{bc}^0) &= (\Gamma_{l,bc}Q)P(\Delta, L_{bc}^0) + (\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0) + \\
&+ (\Gamma_{l,bc}Q)(\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0) + \left(P(\Delta, L_{bc}^0) + (\Gamma_{l,bc}Q)P(\Delta, L_{bc}^0) + \right. \\
&+ (\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0) + (\Gamma_{l,bc}Q)(\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0) \left. \right) \sum_{j=1}^{\infty} (\Gamma_{m,bc}X_*)^j + \\
&+ \left(P(\Delta, L_{bc}^0) + (\Gamma_{l,bc}Q)P(\Delta, L_{bc}^0) + (\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0) + \right. \\
&+ (\Gamma_{l,bc}Q)(\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0) \left. \right) \sum_{j=1}^{\infty} (\Gamma_{l,bc}Q)^j + \\
&+ \left(P(\Delta, L_{bc}^0) + (\Gamma_{l,bc}Q)P(\Delta, L_{bc}^0) + (\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0) + \right. \\
&+ (\Gamma_{l,bc}Q)(\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0) \left. \right) \left(\sum_{j=1}^{\infty} (\Gamma_{m,bc}X_*)^j \right) \left(\sum_{j=1}^{\infty} (\Gamma_{l,bc}Q)^j \right).
\end{aligned}$$

Now we estimate norm of operators $(\Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)$. Using inequality $\|(\Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)\|_2 \leq \|(\Gamma_{bc}Q)P(\Delta, L_{bc}^0)\|_2$ and (3.15), we get

$$\begin{aligned}
\|(\Gamma_{bc}Q)P(\Delta, L_{bc}^0)\|_2^2 &= \sum_{n \in \Omega} \|(\Gamma_{bc}Q)\mathbb{P}_n\|_2^2 \leq \\
&\leq \sum_{n \geq d(\Omega)} \frac{\omega^{4k} \|q\|_2^2}{8\pi^{4k} (2n + \theta)^{4k-4} (2n-1)^2} \leq \frac{\omega^{4k} \|q\|_2^2}{8\pi^{4k} d^{4k-3}(\Omega)}, \quad k > 1.
\end{aligned}$$

Here $d(\Omega) = \min_{n \in \Omega} n$. Therefore,

$$(4.12) \quad \|(\Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)\|_2 \leq \frac{\omega^{2k} \|q\|_2}{2\sqrt{2}\pi^{2k} d^{2k-\frac{3}{2}}(\Omega)}.$$

From the representation of difference $P(\tilde{\Delta}, L_{bc}) - P(\Delta, L_{bc}^0)$ and inequalities (2.23), (4.12), we have

$$\|P(\tilde{\Delta}, L_{bc}) - P(\Delta, L_{bc}^0)\|_2 \leq \frac{M_1}{d^{2k-\frac{3}{2}}(\Omega)},$$

where $M_1 > 0$ is some constant.

In case $bc = dir$ we obtain the same estimates with constant $M_1 > 0$. The proof is carried out analogously.

Now we prove inequality (1.10). Put $\mathcal{U} = \Gamma_{l,bc}Q$. Using (4.11), we obtain

$$\begin{aligned}
P(\tilde{\Delta}, L_{bc}) &= \\
&= (I + \Gamma_{l,bc}Q)(I + \Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0)(I + \Gamma_{m,bc}X_*)^{-1}(I + \Gamma_{l,bc}Q)^{-1} = \\
&= (I + \Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)(I + \Gamma_{m,bc}X_*)^{-1}(I + \Gamma_{l,bc}Q)^{-1} + \\
&+ (I + \Gamma_{l,bc}Q)(\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0)(I + \Gamma_{m,bc}X_*)^{-1}(I + \Gamma_{l,bc}Q)^{-1} = \\
&= (I + \Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)(I + \Gamma_{l,bc}Q)^{-1} + \\
&+ (I + \Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)((I + \Gamma_{m,bc}X_*)^{-1} - I)(I + \Gamma_{l,bc}Q)^{-1} + \\
&+ (I + \Gamma_{l,bc}Q)(\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0)(I + \Gamma_{m,bc}X_*)^{-1}(I + \Gamma_{l,bc}Q)^{-1} = \\
&= (I + \Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)(I + \Gamma_{l,bc}Q)^{-1} + \\
&+ (I + \Gamma_{l,bc}Q)P(\Delta, L_{bc}^0). \\
&\cdot (\Gamma_{m,bc}X_*) \left(\sum_{j=0}^{\infty} (-1)^{j+1} (\Gamma_{m,bc}X_*)^j \right) (I + \Gamma_{l,bc}Q)^{-1} + \\
&+ (I + \Gamma_{l,bc}Q)(\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0)(I + \Gamma_{m,bc}X_*)^{-1}(I + \Gamma_{l,bc}Q)^{-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|P(\tilde{\Delta}, L_{bc}) - (I + \Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)(I + \Gamma_{l,bc}Q)^{-1}\|_2 &\leq \\
&\leq C_8(\|P(\Delta, L_{bc}^0)(\Gamma_{m,bc}X_*)\|_2 + \|(\Gamma_{m,bc}X_*)P(\Delta, L_{bc}^0)\|_2),
\end{aligned}$$

where $C_8 > 0$ is some constant. Using (2.23), we get

$$\|P(\tilde{\Delta}, L_{bc}) - (I + \Gamma_{l,bc}Q)P(\Delta, L_{bc}^0)(I + \Gamma_{l,bc}Q)^{-1}\|_2 \leq \frac{M_2}{d^{2k-1}(\Omega)},$$

where $M_2 > 0$ is some constant. The Theorem is proved.

Corollary 1.9 follows immediately from Theorem 1.8.

Finally, we prove theorem on asymptotic behavior of analytic semi-group of operators.

Proof of theorem 1.10. We will use the following statement (see [32, Chapter 1, Section 6, problem 2]).

Lemma 4.2. *Let \mathcal{A} be a matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then*

$$e^{\mathcal{A}t} = e^{\frac{a+d}{2}t} \left(\text{ch}(\rho t) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \frac{\text{sh}(\rho t)}{\rho} \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix} \right),$$

where $\rho = \sqrt{\frac{(a-d)^2}{2} + bc}$.

By Theorem 3.6, the operator L_{bc} , $bc \in \{per, ap, dir\}$, is similar to $L_{bc}^0 - J_{m,bc}X_*$. Since $J_{m,bc}X_*$ is bounded operator and $-L_{bc}^0$ is sectorial operator, then operator $-L_{bc}^0 + J_{m,bc}X_*$ is sectorial (see [33, Theorem 1.3.2]). Hence, the operator $-L_{bc}$ is sectorial (see also [7, Section 5]) and generates an analytic semigroup of operators $T : \mathbb{R}_+ \rightarrow \text{End } L_2[0, \omega]$. By Theorem 3.6, there exists a number $m \in \mathbb{Z}_+$, such that this semigroup is similar to semigroup $\tilde{T} : \mathbb{R}_+ \rightarrow \text{End } L_2[0, \omega]$ of the form

$$\tilde{T}(t) = T_{(m)}(t) \oplus T^{(m)}(t), \quad t \in \mathbb{R}_+.$$

It is acting in $L_2[0, \omega] = \mathcal{H}_{(m)} \oplus \mathcal{H}^{(m)}$, where $\mathcal{H}_{(m)} = \text{Im } \mathbb{P}_{(m)}$, $\mathcal{H}^{(m)} = \text{Im } (I - \mathbb{P}_{(m)})$ for $bc \in \{per, ap\}$ and $\mathcal{H}_{(m)} = \text{Im } P_{(m)}$, $\mathcal{H}^{(m)} = \text{Im } (I - P_{(m)})$ for $bc = dir$.

If $bc = dir$, then the semigroup $T^{(m)} : \mathbb{R}_+ \rightarrow \text{End } \mathcal{H}^{(m)}$ has the representation

$$T^{(m)}(t)x = \sum_{s \geq m+1} e^{-\tilde{\lambda}_{s,dir}t} P_{s,dir}x, \quad x \in L_2[0, \omega],$$

where $\tilde{\lambda}_{s,dir}$ is defined in (1.7).

To obtain the asymptotic behavior of the semigroup in case $bc \in \{per, ap\}$, we need to apply Lemma 4.2. Put $\mathcal{A} = \left(\frac{\pi(2n+\theta)}{\omega}\right)^{2k} I_n - \mathcal{A}_n$, $\theta \in \{0, 1\}$, $k > 1$. Here \mathcal{A}_n is a matrix of a restriction of operator $J_{m,bc}(\mathbb{P}_n X_* \mathbb{P}_n)$ on subspace \mathcal{H}_n in the basis $e_{-n-\theta}$, e_n . Using representation (4.2) and estimates (4.10), we get (1.11). From Theorems 1.1 and 1.2 we obtain all assertions of this theorem. The Theorem is proved.

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